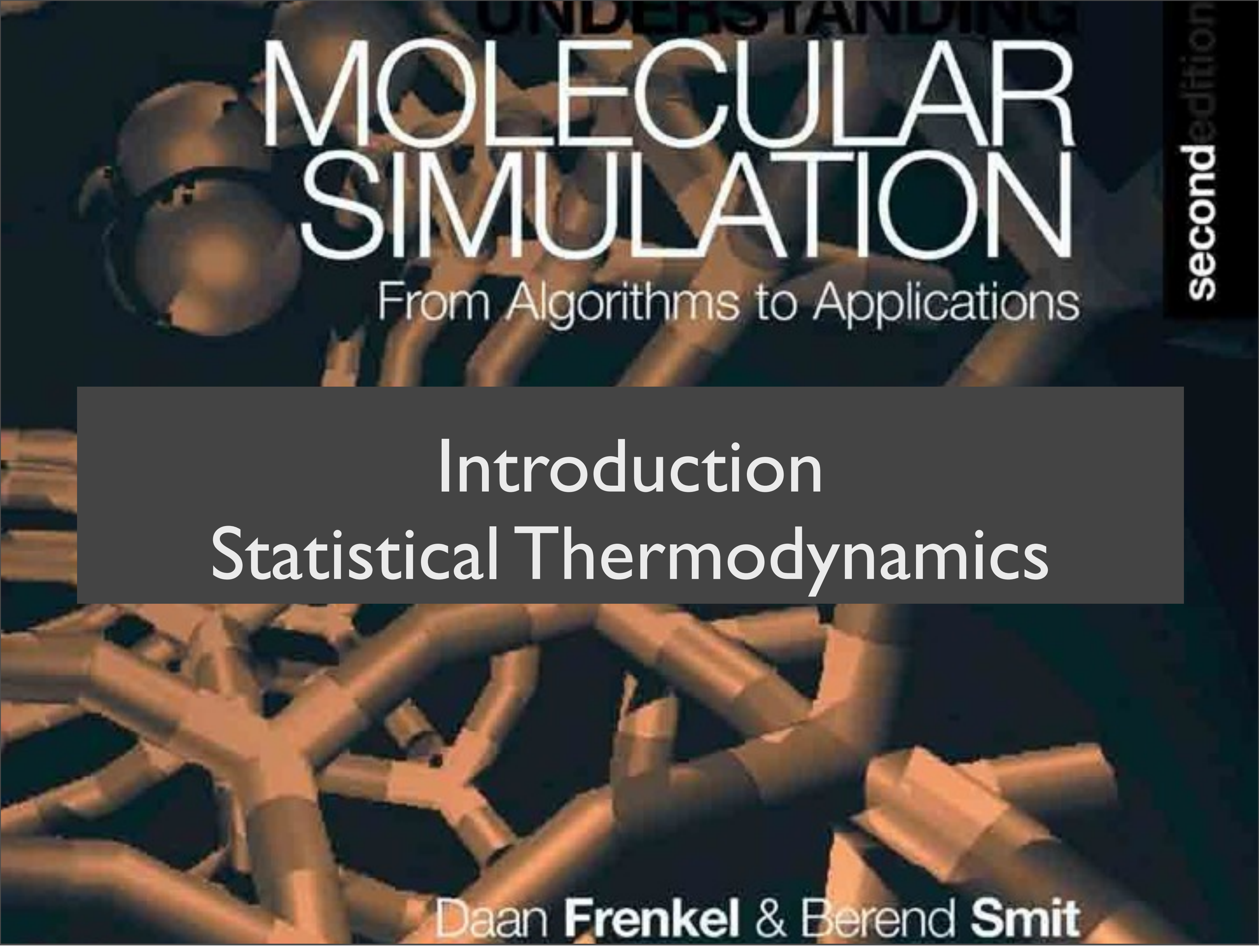
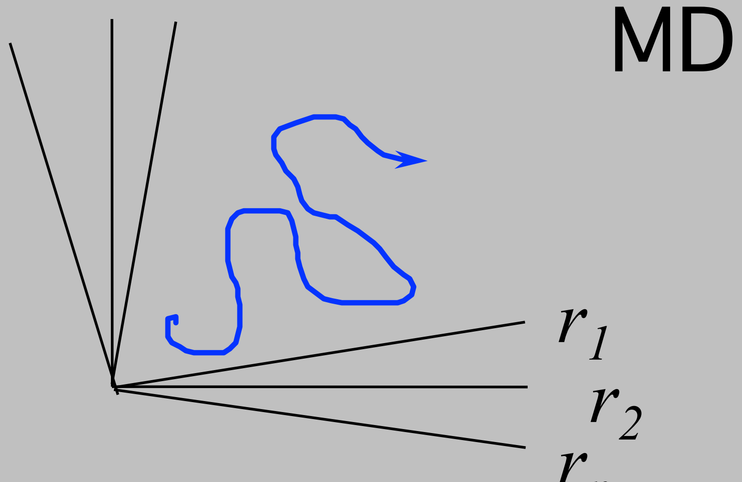


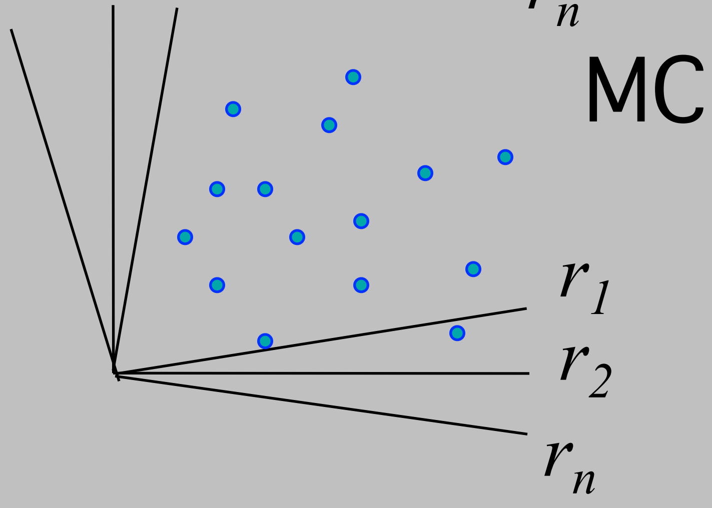
Introduction
Statistical Thermodynamics

Molecular Simulations

- Molecular dynamics: solve equations of motion



- Monte Carlo: importance sampling



Algorithm 1 (Basic Metropolis Algorithm)

```
  PROGRAM mc                                 basic Metropolis algorithm

  do icycl=1,ncycl                           perform ncycl MC cycles
    call mcmove                              displace a particle
    if (mod(icycl,nsamp).eq.0)
+       call sample                          sample averages
  enddo
  end
```

Comments to this algorithm:

1. *Subroutine* `mcmove` *attempts to displace a randomly selected particle (see Algorithm 2).*
2. *Subroutine* `sample` *samples quantities every* `nsamp`*th cycle.*

Algorithm 2 (Attempt to Displace a Particle)

```
  SUBROUTINE mcmove               attempts to displace a particle

  o=int(ranf()*npart)+1           select a particle at random
  call ener(x(o),eno)             energy old configuration
  xn=x(o)+(ranf()-0.5)*delx       give particle random displacement
  call ener(xn,enn)               energy new configuration
  if (ranf().lt.exp(-beta         acceptance rule (3.2.1)
+    *(enn-eno)) x(o)=xn          accepted: replace x(o) by xn
  return
  end
```

Comments to this algorithm:

1. *Subroutine* `ener` *calculates the energy of a particle at the given position.*
2. *Note that, if a configuration is rejected, the old configuration is retained.*
3. *The* `ranf()` *is a random number uniform in* $[0, 1]$.

Questions

What is the desired distribution?

- How can we prove that this scheme generates the desired distribution of configurations?
- Why make a random selection of the particle to be displaced?
- Why do we need to take the old configuration again?
- How large should we take: `delx?`

Outline

Rewrite History

- Atoms first! Thermodynamics last!

Thermodynamics

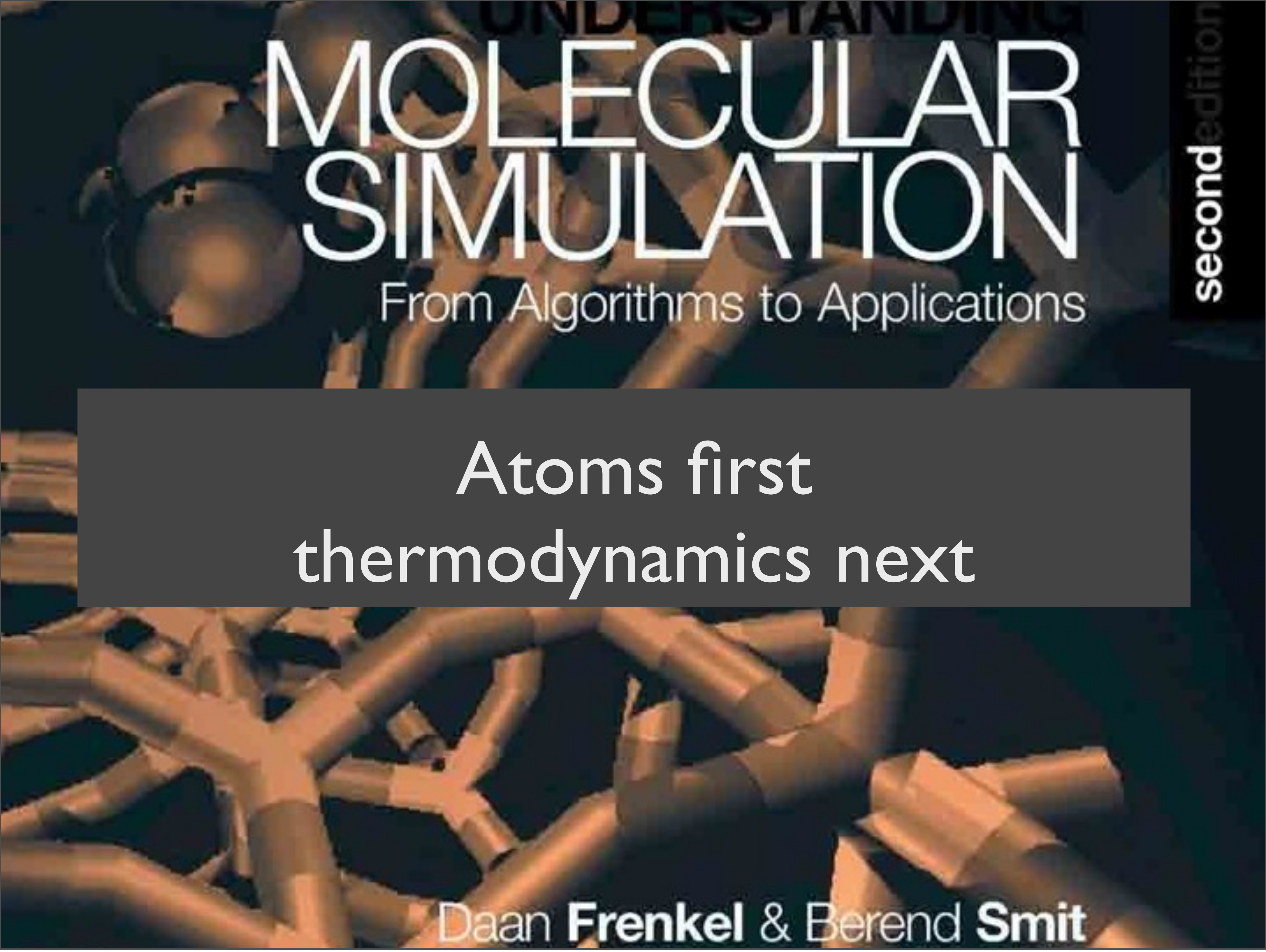
- First law: conservation of energy
- Second law: in a closed system entropy increase and takes its maximum value at equilibrium

System at constant temperature and volume

- Helmholtz free energy decrease and takes its minimum value at equilibrium

Other ensembles:

- Constant pressure
- grand-canonical ensemble



UNDERSTANDING

MOLECULAR SIMULATION

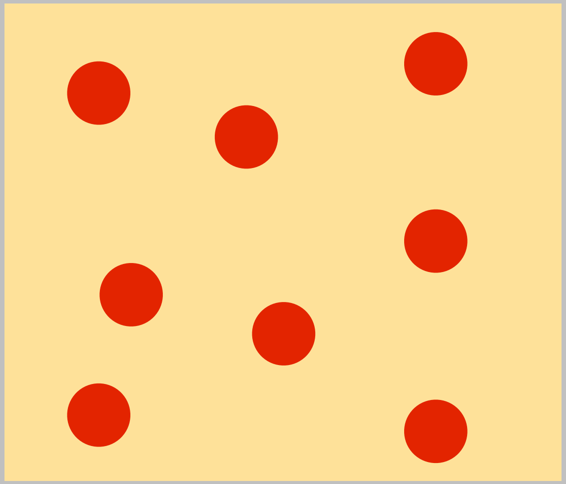
From Algorithms to Applications

secondedition

Daan **Frenkel** & Berend **Smit**

Atoms first
thermodynamics next

A box of particles



We have given the particles an intermolecular potential

Newton: equations of motion

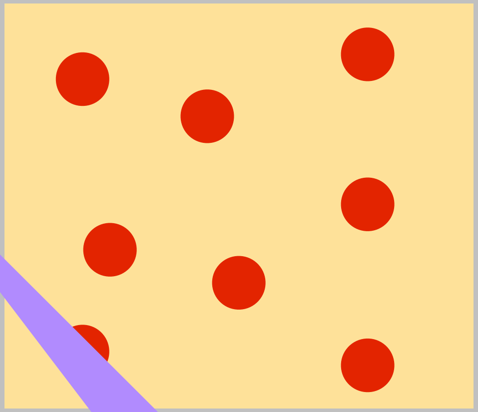
$$\mathbf{F}(r) = -\nabla u(r)$$

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}(r)$$

Conservation of total energy

Phase space

Thermodynamics: N,V,E

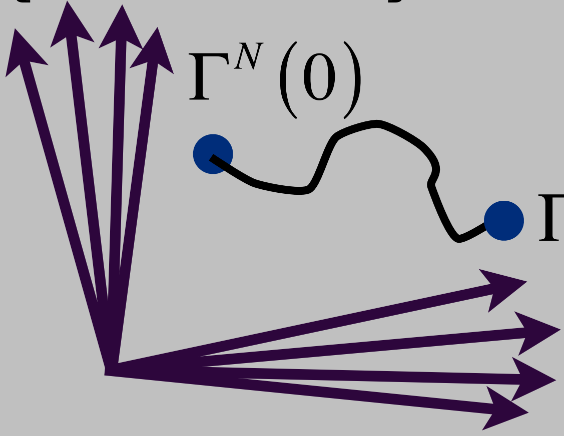


Why this one?

Molecular:

$$\Gamma^N = \{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N, \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N\}$$

point in phase space



$\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N\}$

$\Gamma^N(0)$

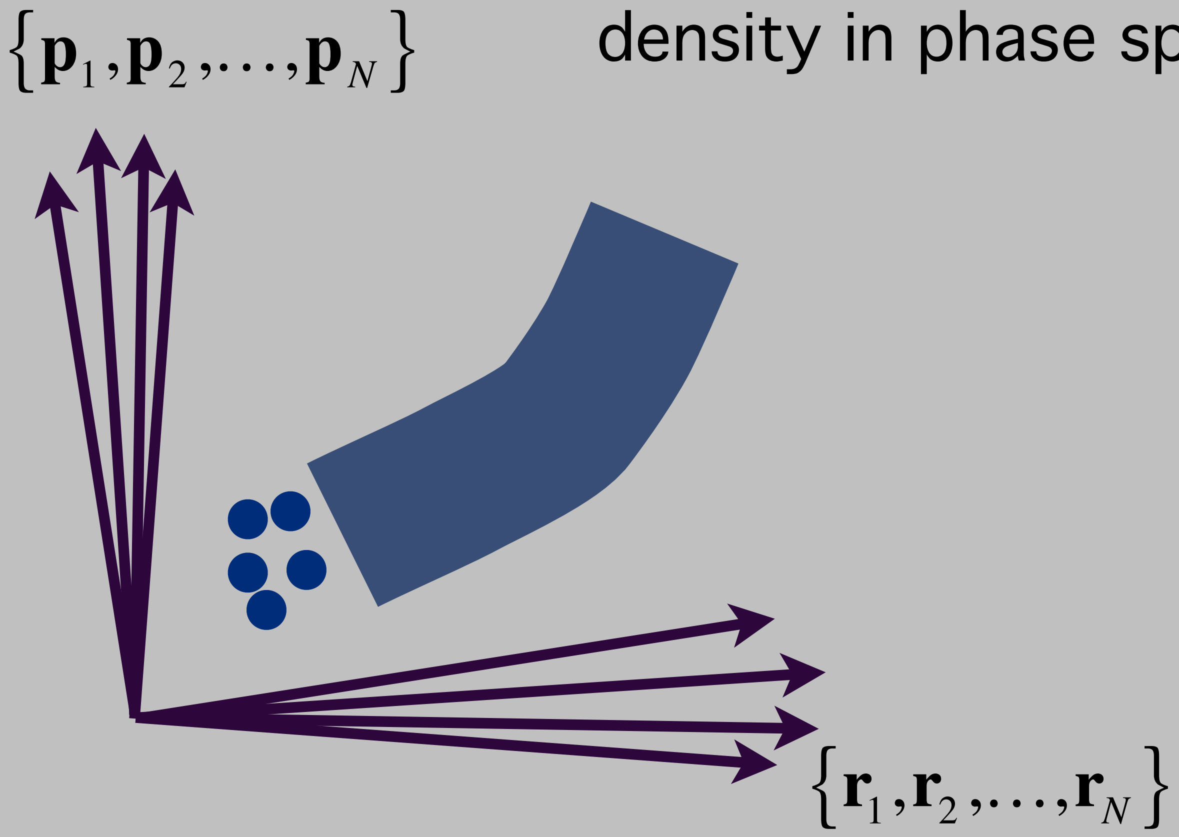
$\Gamma^N(t)$

$\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N\}$

trajectory: classical mechanics

All trajectories with the same initial total energy should describe the same thermodynamic state

These trajectories define a probability density in phase space



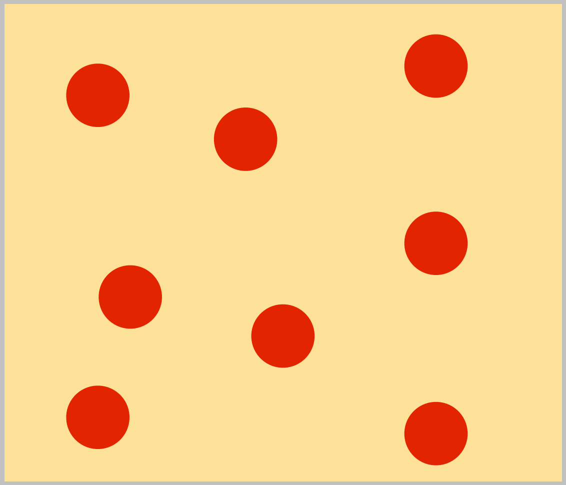
$\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N\}$

$\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N\}$

Intermezzo 1: phase rule

- Question: explain the phase rule?
- Phase rule: F=2-P+C
    - F: degrees of freedom
    - P: number of phases
    - C: number of components
- Why the 2?

Making a gas



What do we need to specify to fully define a thermodynamic system?

1. Specify the volume V
2. Specify the number of particles N
3. Give the particles:
   - initial positions
   - initial velocities

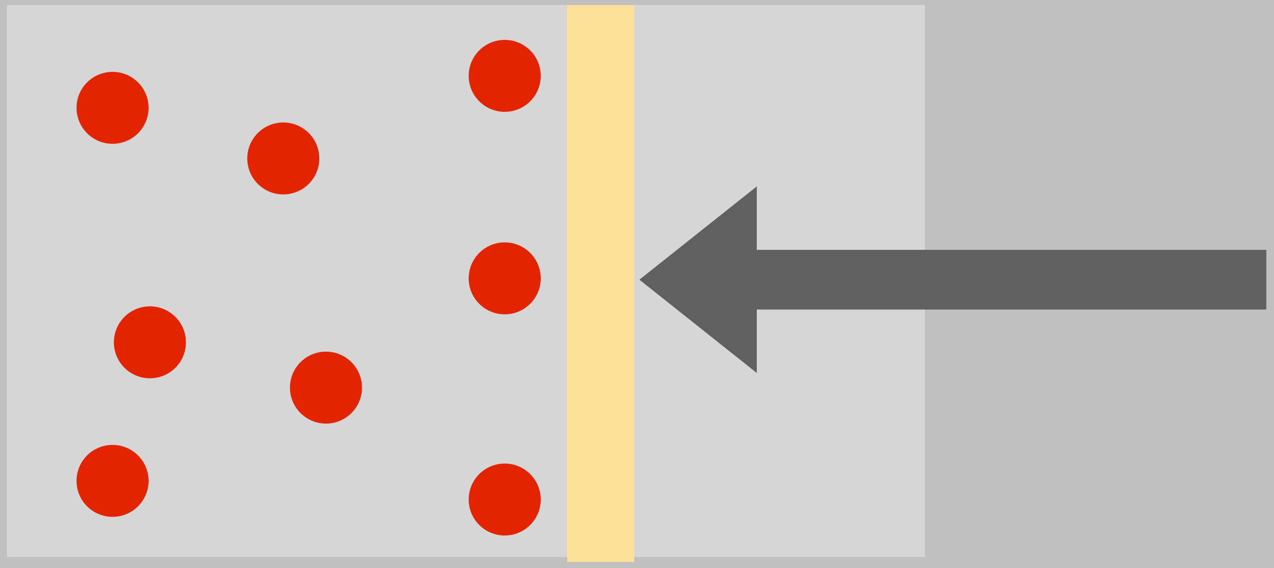
More we cannot do: Newton takes over!

System will be at constant: N,V,E

(micro-canonical ensemble)

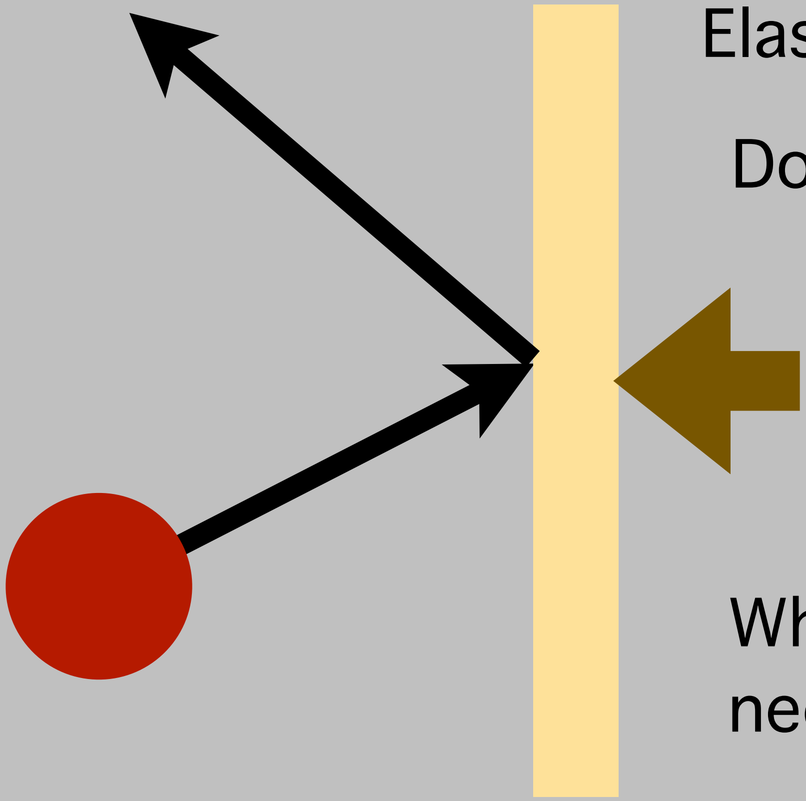
Pressure

What is the force I need to apply to prevent the wall from moving?



How much work I do?

Collision with a wall



Elastic collisions:

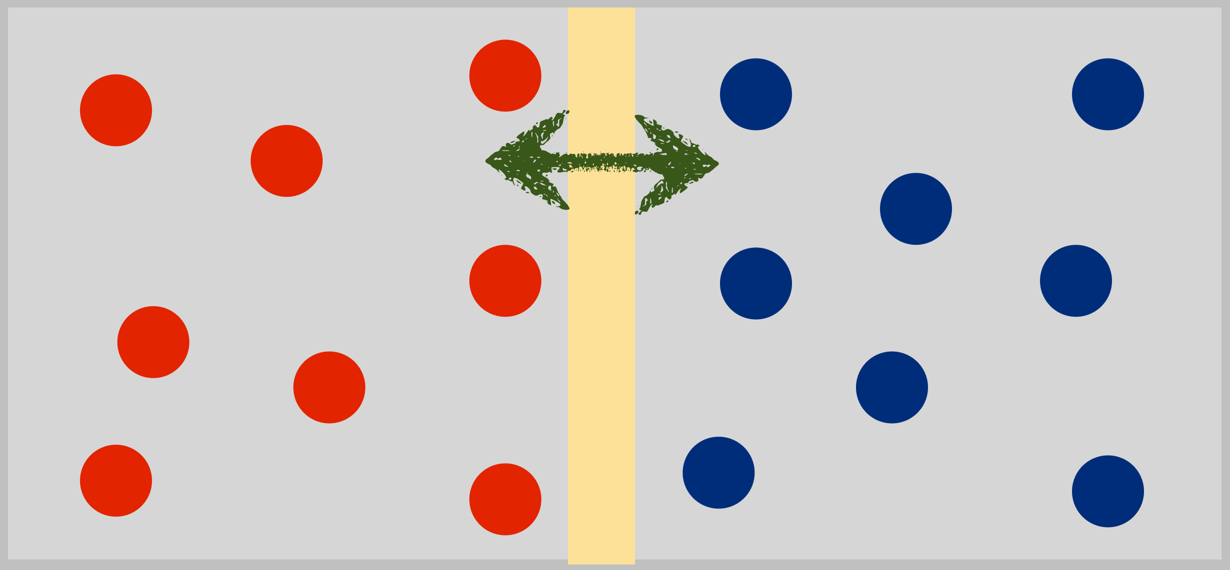
Does the energy change?

What is the force that we need to apply on the wall?

Pressure

- one particle: $2\ m\ v_x$
- # particles: $\rho\ A\ v_x$
- 50% is the positive directions: 0.5
- $P\ A = F = \rho\ A\ m\ v_x^2$
- Kinetic energy: $U_K = \frac{1}{2}\ m\ v^2 = \frac{3}{2}\ k_B\ T$
    - (we define temperature)
- Pressure: $P\ V = N\ k_B\ T$

Experiment (1)



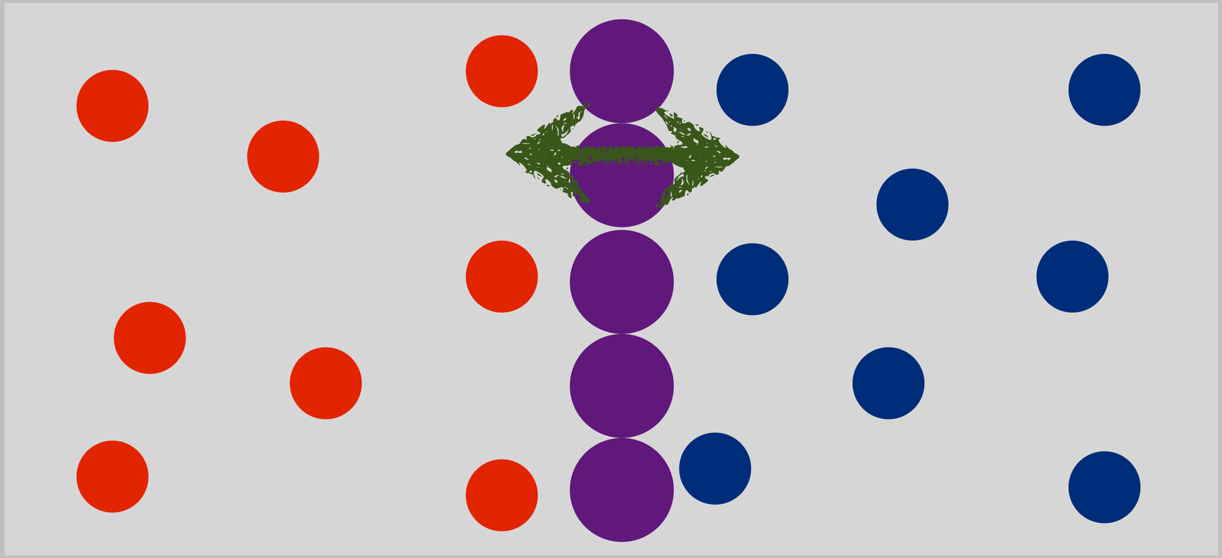
NVE$_1$

NVE$_2$

$E_1 > E_2$

What will the moveable wall do?

Experiment (2)



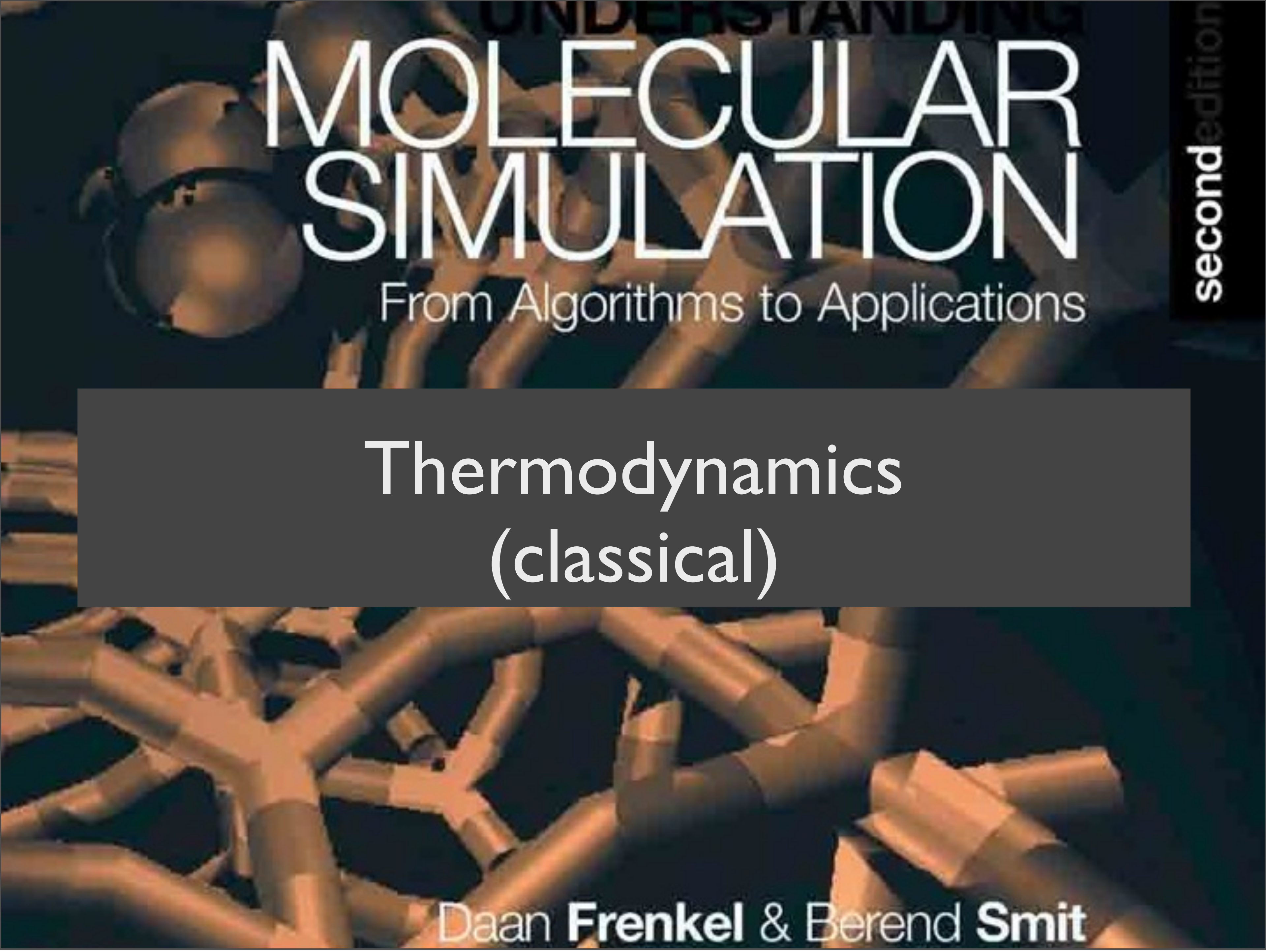
NVE_1 NVE_2 $E_1 > E_2$

Now the wall are heavy molecules

What will the moveable wall do?

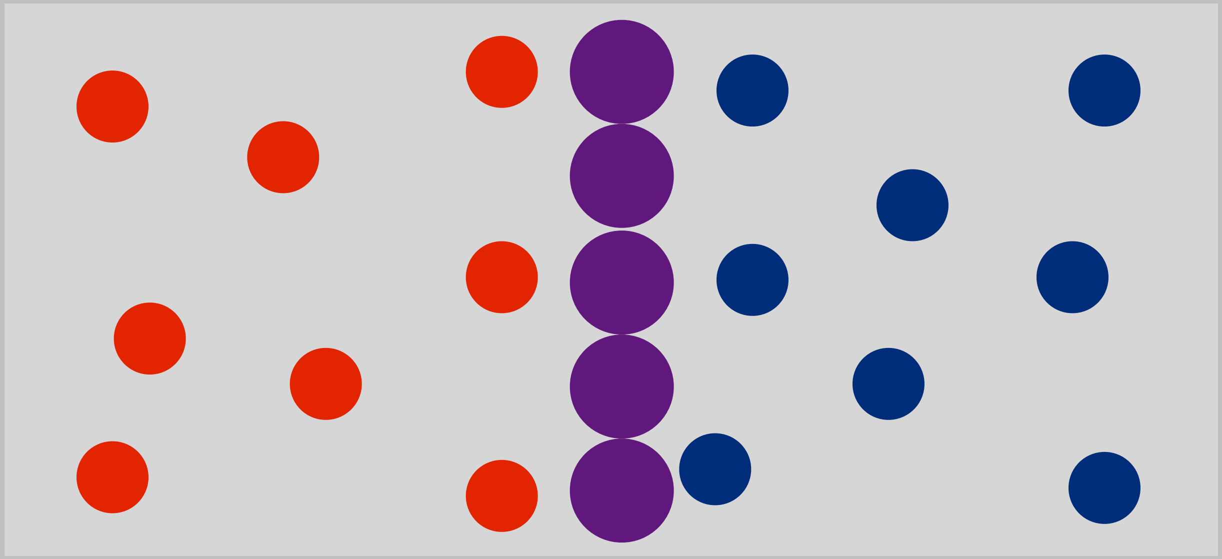
Newton + atoms

- We have a natural formulation of the first law
- We have discovered pressure
- We have discovered another equilibrium properties related to the total energy of the system



Thermodynamics (classical)

Experiment



NVE$_1$ NVE$_2$ $E_1 > E_2$

The wall can move and exchange energy:

what determines equilibrium ?

Classical Thermodynamics

- 1st law of Thermodynamics
  - Energy is conserved
- 2nd law of Thermodynamics
  - Heat spontaneously flows from hot to cold

Classical Thermodynamics

Carnot: Entropy difference between two states:

$$\Delta S = S_B - S_A = \int_A^B \frac{dQ_{rev}}{T}$$

Using the first law we have:

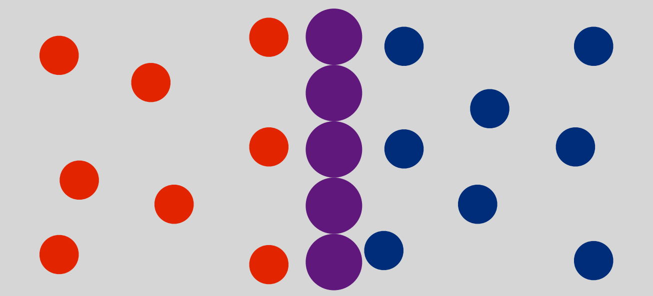
$$\Delta U = Q + W$$

If we carry out a reversible process, we have for any point along the path

$$dU = TdS + dW$$

If we have work by a expansion of a fluid

$$\boxed{dU = TdS - pdV}$$



Let us look at the <u>very initial</u> stage

dq is so small that the temperatures of the two systems do not change

For system H $$\mathrm{d}S_{\mathrm{H}} = -\frac{\mathrm{dq}}{\mathrm{T_H}}$$

For system L $$\mathrm{d}S_{\mathrm{L}} = \frac{\mathrm{dq}}{\mathrm{T_L}}$$

Hence, for the total system

$$\mathrm{d}S = \mathrm{d}S_{\mathrm{L}} + \mathrm{d}S_{\mathrm{H}} = \mathrm{dq}\left(\frac{1}{\mathrm{T_L}} - \frac{1}{\mathrm{T_H}}\right)$$

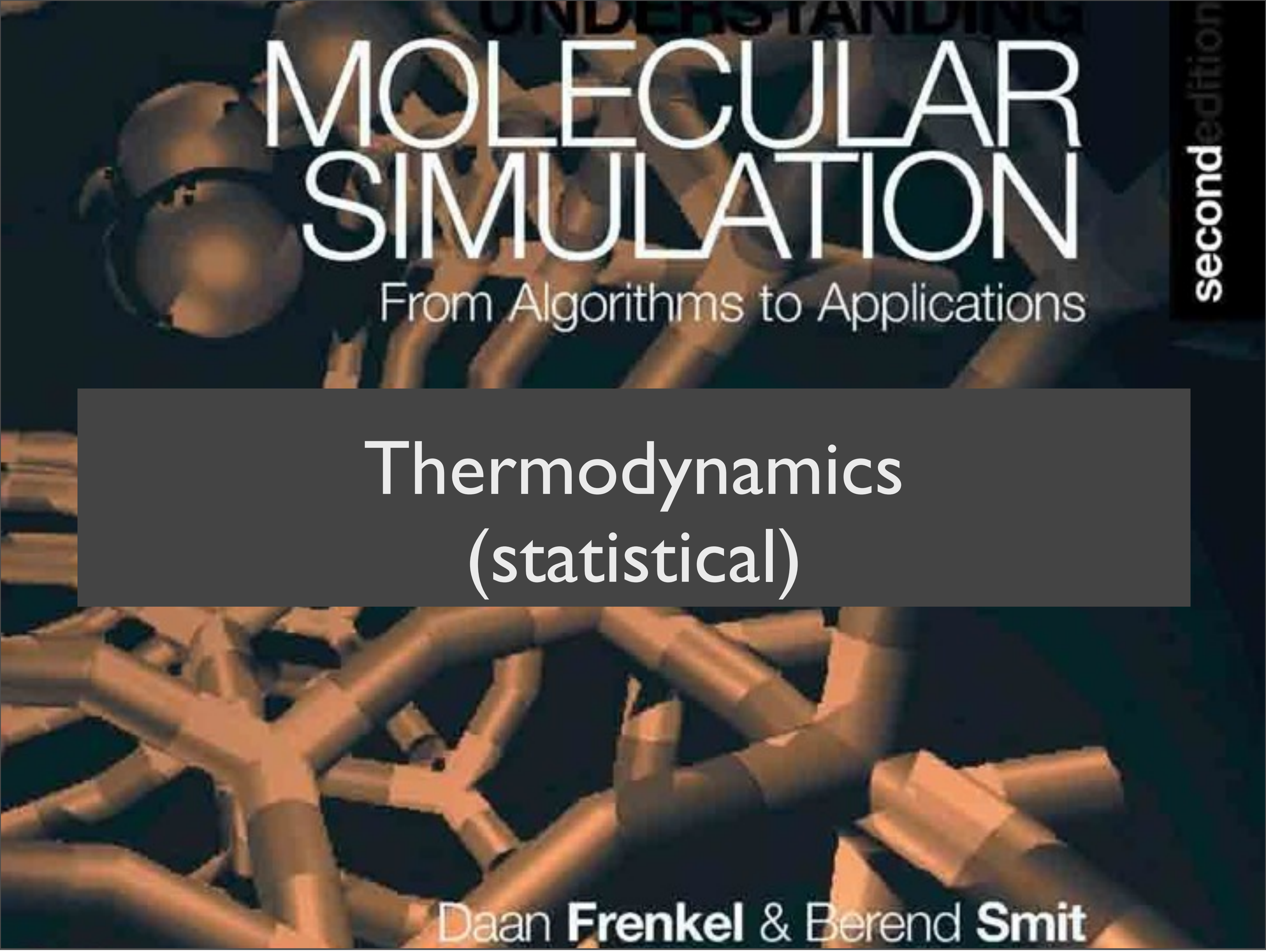
Heat goes from warm to cold: or if dq > 0 then $T_H > T_L$

This gives for the entropy change: $dS > 0$

Hence, the entropy increases until the two temperatures are equal

Question

- Thermodynamics has a sense of time, but not Newton's dynamics
    - Look at a water atoms in reverse
    - Look at a movie in reverse
- When do molecules know about the arrow of time?



UNDERSTANDING
MOLECULAR
SIMULATION
From Algorithms to Applications
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Thermodynamics
(statistical)

Statistical The ics

Basic assumpti

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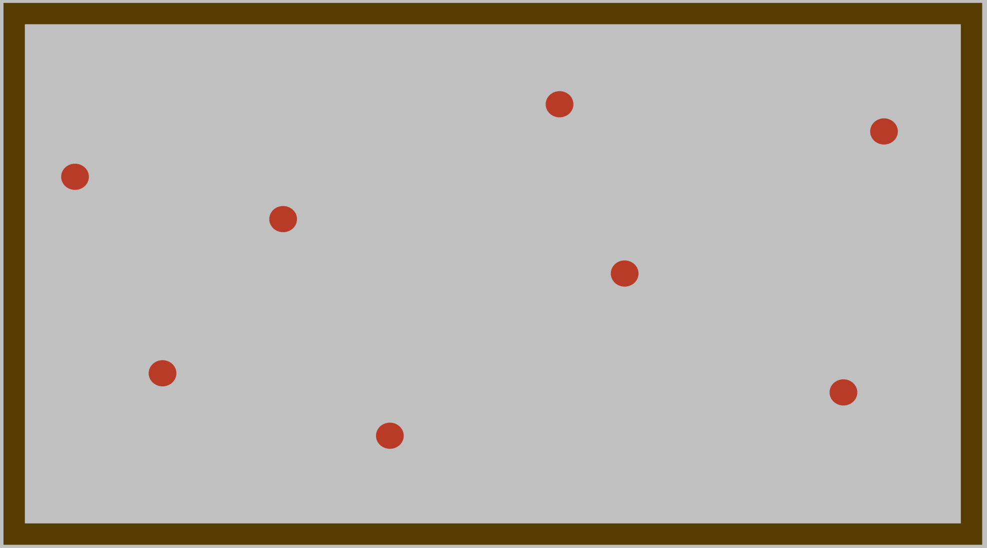
··· but classical thermodynamics is based on laws

Ideal gas

Let us again make an ideal gas



We select:
(1) N particles,
(2) Volume V,
(3) initial velocities
+ positions

This fixes; V/n, U/n

Basic assumption

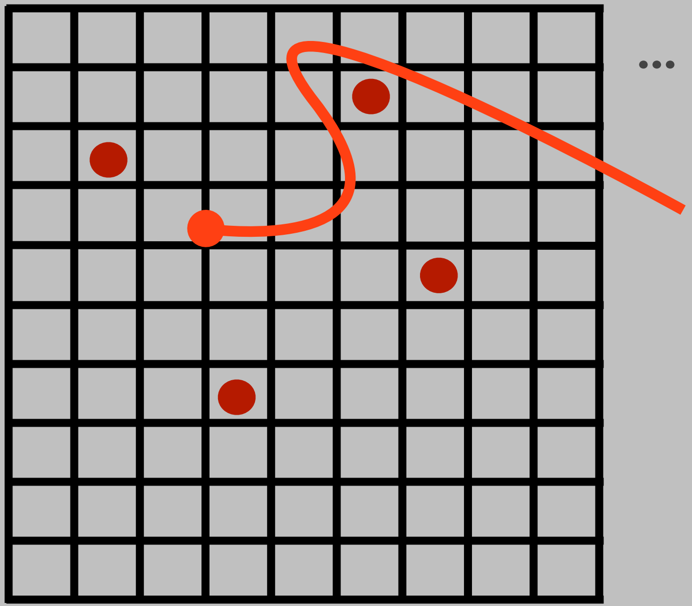
For an isolated system any microscopic configuration is equally likely

What is the probability to find this configuration?

The system has the same kinetic energy!!

Our basic assumption must be seriously wrong!

... but are we doing the statistics correctly?



… lets look at our statistics correctly

What is the probability to find this configuration?

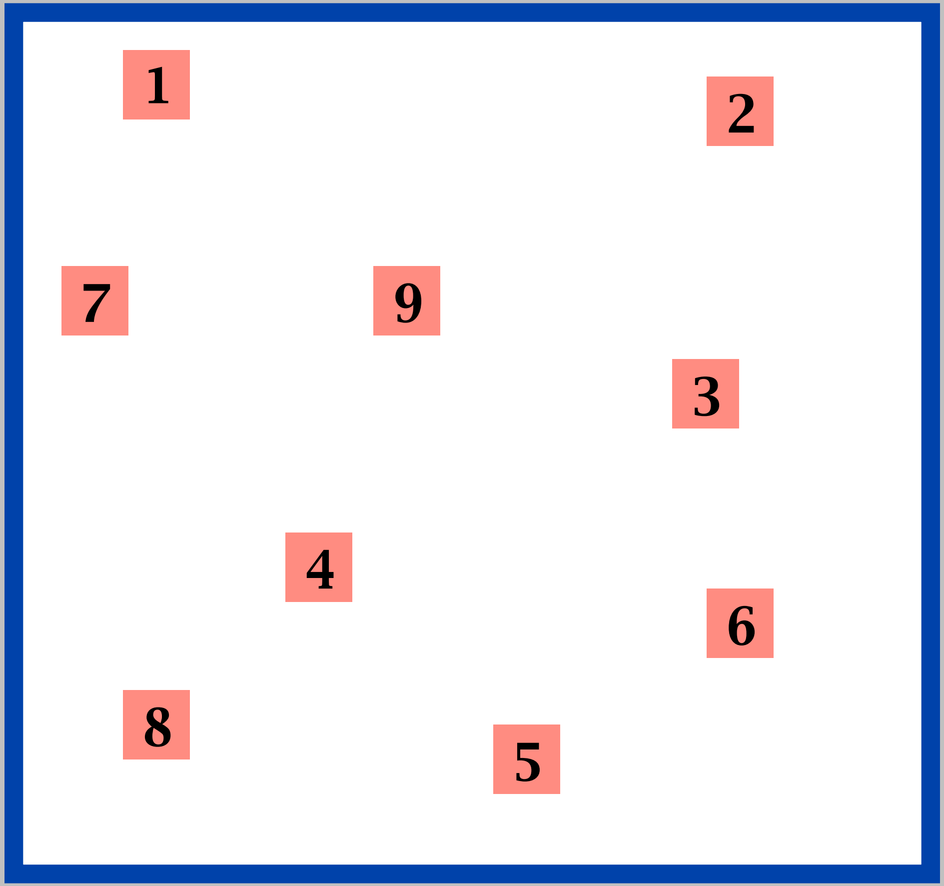
Basic assumption:

$$\mathcal{P} = \frac{1}{\text{total \# of configurations}}$$

number 1 can be put in M positions, number 2 at M positions, etc

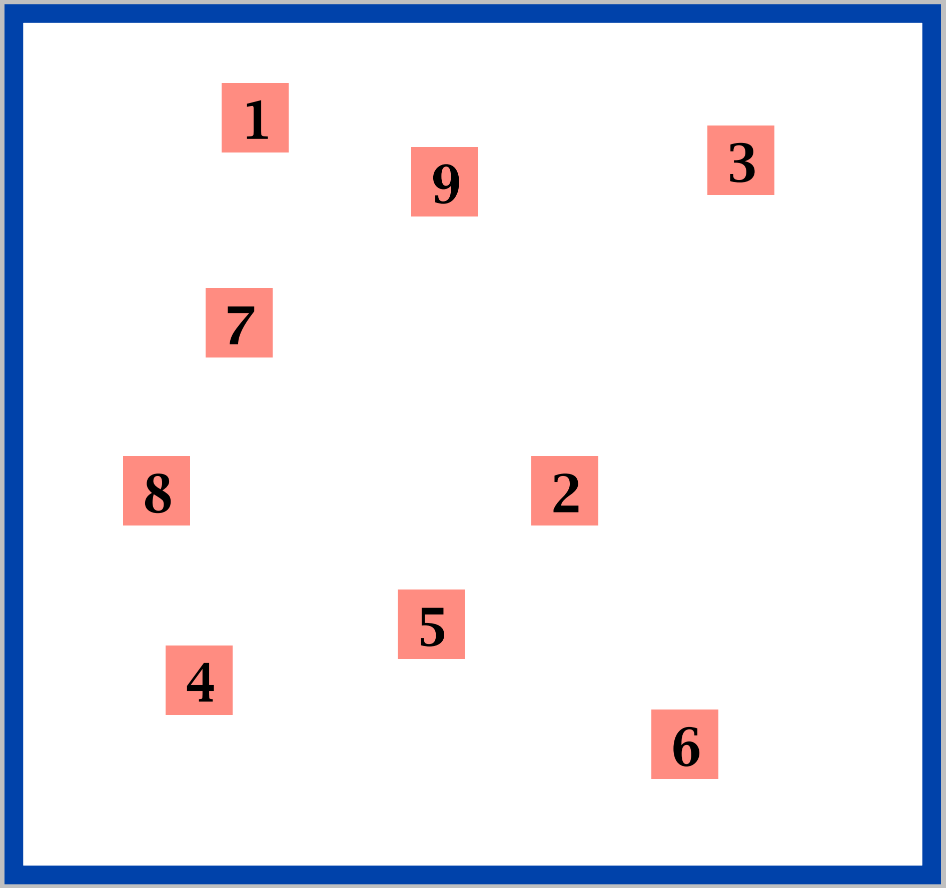
Total number of configurations: M^{N} with $\mathrm{M} = \frac{\mathrm{V}}{\mathrm{dr}}$

the larger the volume of the gas the more configurations



What is the probability to find the 9 molecules exactly at these 9 positions?

$$\left(\frac{\Delta V}{V}\right)^N$$



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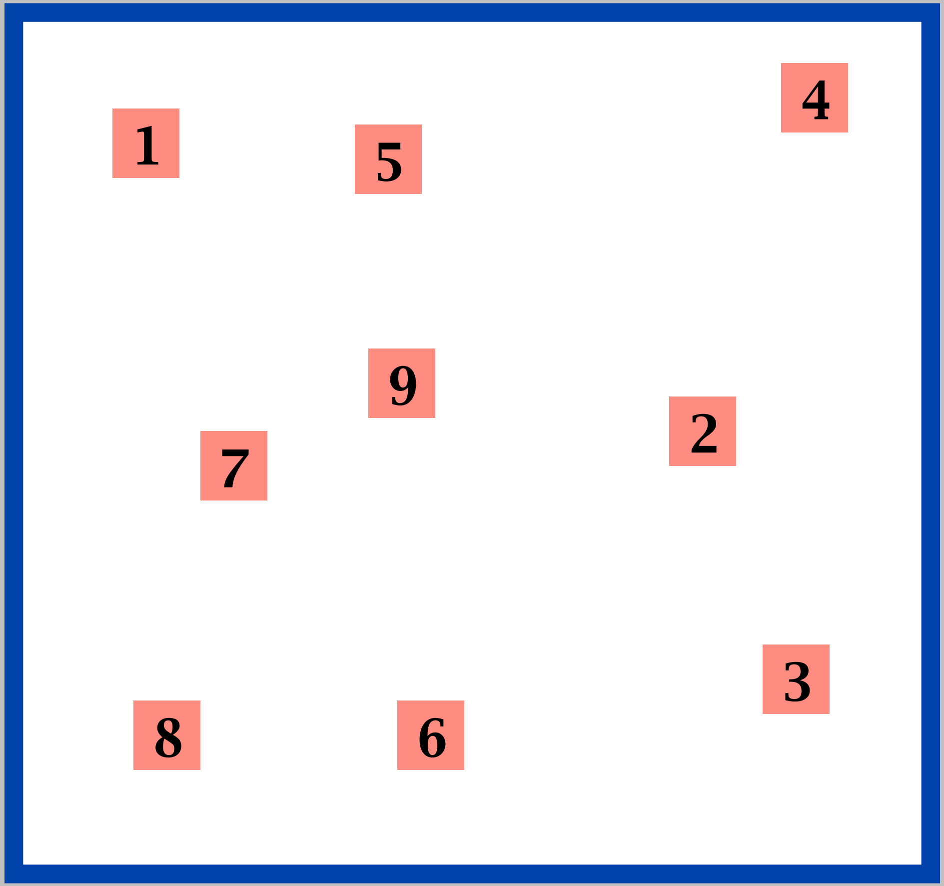

What is the probability to find the 9 molecules exactly at these 9 positions?

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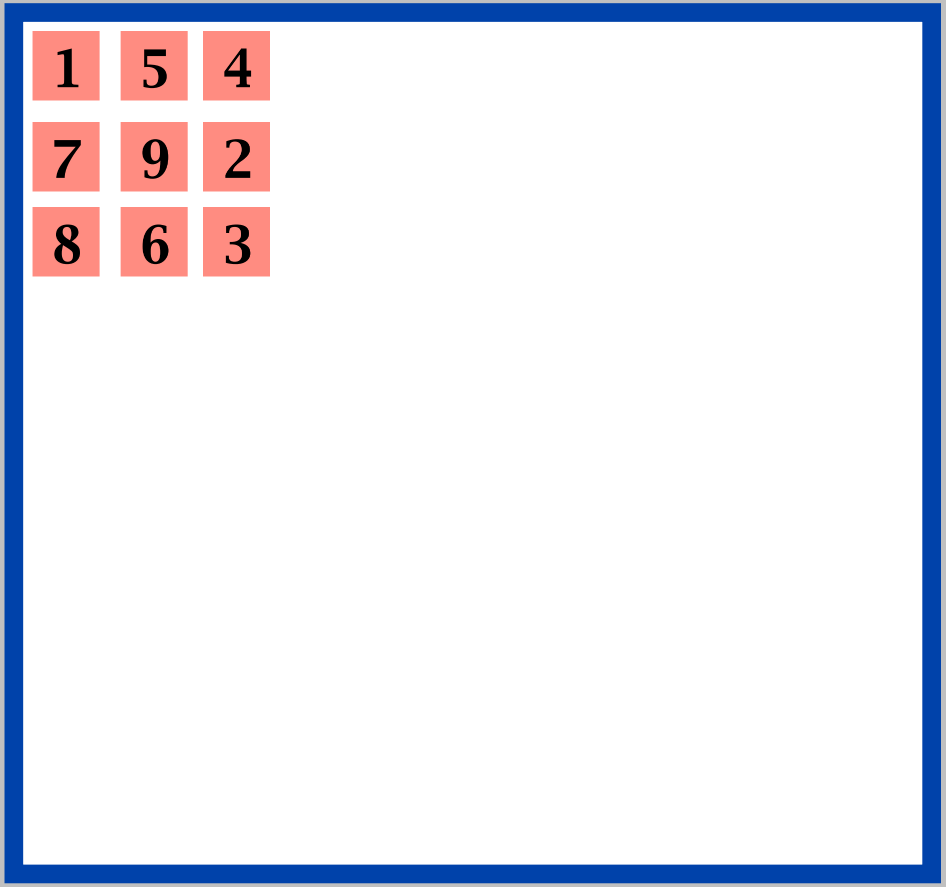
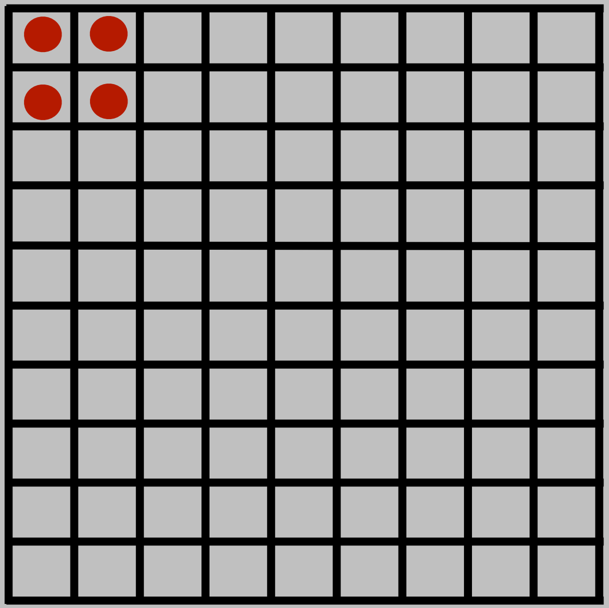


What is the probability to find the 9 molecules exactly at these 9 positions?

$$\boxed{\left(\frac{\Delta V}{V}\right)^N}$$

Question

- Is it safe to be in this room?



What is the probability to find this configuration?

exactly equal as to any other configuration!!!!!!

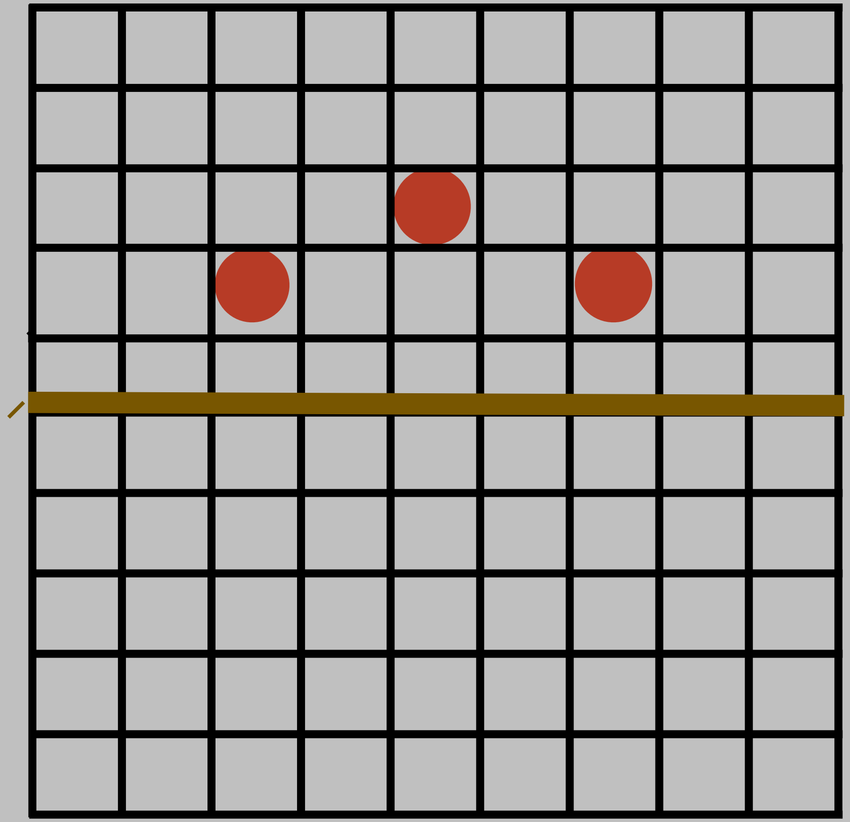
This is reflecting the microscopic reversibility of Newton's equations of motion. A microscopic system has no "sense" of the direction of time

Are we asking the **right question**?

Are we asking the right question?

These are microscopic properties; **no irreversibility**

Thermodynamic is about macroscopic properties:



Measure densities: what is the probability that we have all our N gas particle in the upper half?

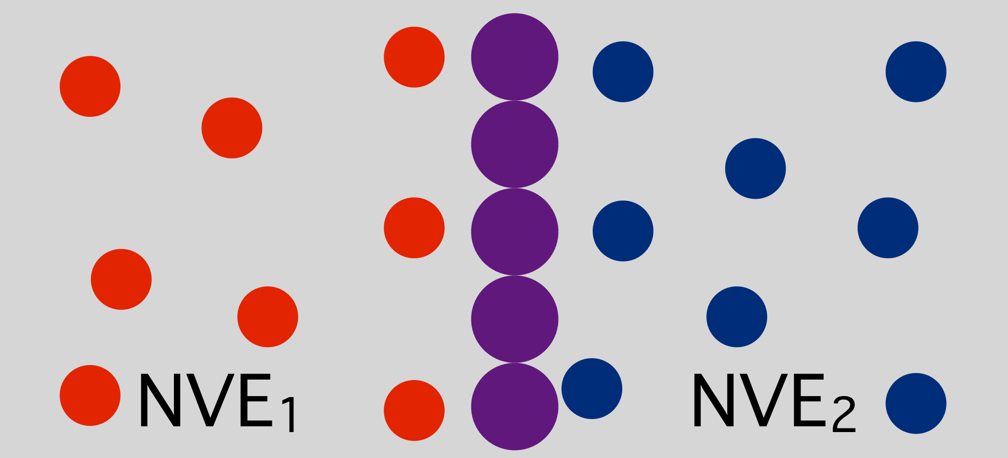
N	P(empty)
1	0.5
2	0.5 x 0.5
3	0.5 x 0.5 x 0.5
1000	10^{-301}

Summary

- On a microscopic level all configurations are equally likely
- On a macroscopic level; as the number of particles is extremely large, the probability that we have a fluctuation from the average value is extremely low
- Let us quantify these statements

Basic assumption

$E_1 > E_2$



Let us look at one of our examples; let us assume that the total system is isolate but heat can flow between 1 and 2.

All micro states will be equally likely!

... but the number of micro states that give an particular energy distribution $(E_1,E\text{-}E_1)$ not ...

... so, we observe the most likely one ...

In a macroscopic system we will observe the most likely one

$$\mathcal{P}(E_1, E_2) = \frac{\mathcal{N}_1(E_1) \times \mathcal{N}_2(E - E_1)}{\sum_{E_1=0}^{E_1=E} \mathcal{N}_1(E_1) \times \mathcal{N}_2(E - E_1)}$$

The summation only depends on the total energy:

$$\mathcal{P}(E_1, E_2) = C \times \mathcal{N}_1(E_1) \times \mathcal{N}_2(E - E_1)$$

$$\ln \mathcal{P}(E_1, E_2) = \ln C + \ln \mathcal{N}_1(E_1) + \ln \mathcal{N}_2(E - E_1)$$

We need to find the maximum

$$\frac{d \ln \mathcal{P}(E_1, E_2)}{dE_1} = \frac{d \ln \mathcal{N}(E_1, E_2)}{dE_1} = 0$$

$$\frac{d\left[\ln \mathcal{N}_1(E_1) + \ln \mathcal{N}_2(E - E_1)\right]}{dE_1} = 0$$

We need to find the maximum

$$\frac{\mathrm{d}\left[\ln \mathcal{N}_1(E_1) + \ln \mathcal{N}_2(E - E_1)\right]}{\mathrm{d}E_1} = 0$$

$$\frac{\mathrm{d}\ln \mathcal{N}_1(E_1)}{\mathrm{d}E_1} = -\frac{\mathrm{d}\ln \mathcal{N}_2(E - E_1)}{\mathrm{d}E_1}$$

As the total energy is constant

$$E_2 = E - E_1$$

$$\mathrm{d}E_1 = -\mathrm{d}(E - E_1) = -\mathrm{d}E_2$$

Which gives as equilibrium condition:

$$\frac{\mathrm{d}\ln \mathcal{N}_1(E_1)}{\mathrm{d}E_1} = \frac{\mathrm{d}\ln \mathcal{N}_2(E_2)}{\mathrm{d}E_2}$$

Let us define a property
(almost S, but not quite) :

$$S^* = \ln \mathfrak{N}(E)$$

Equilibrium if:

$$\frac{d\ln \mathfrak{N}_1(E_1)}{dE_1} = \frac{d\ln \mathfrak{N}_2(E_2)}{dE_2}$$

or

$$\left(\frac{\partial S_1^*}{\partial E_1}\right)_{N_1,V_1} = \left(\frac{\partial S_2^*}{\partial E_2}\right)_{N_2,V_2}$$

And for the total system:

$$S^* = S_1^* + S_2^*$$

For a system at constant energy, volume and number of particles the S^* increases until it has reached its maximum value at equilibrium

What is this magic property S^*?

Defined a property S^* (that is almost S):

$$\begin{aligned} S^*(E_1, E-E_1) &= \ln \aleph(E_1, E-E_1) \\ &= \ln \aleph_1(E_1) + \ln \aleph_2(E-E_1) \\ &= S_1^*(E_1) + S_2^*(E-E_1) \end{aligned}$$

Why is maximizing S^* the same as maximizing N?

The logarithm is a monotonically increasing function.

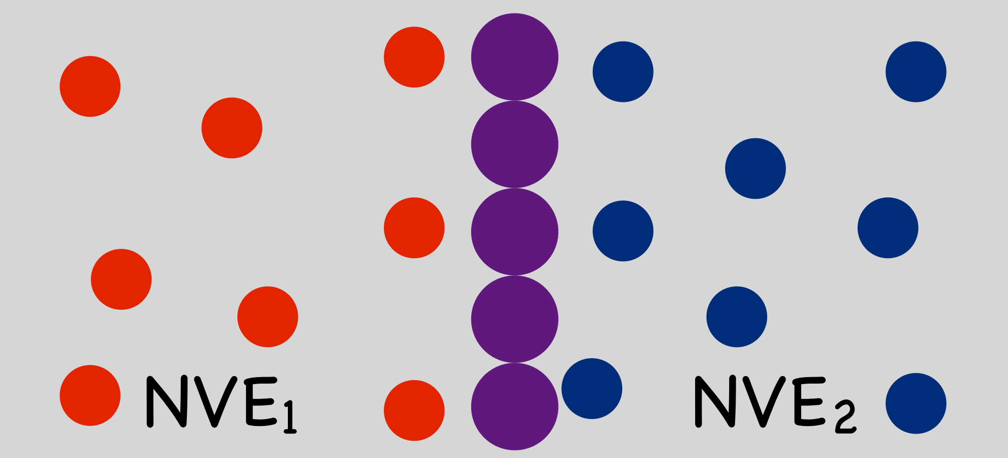
Why else is the logarithm a convenient function?

Makes S* additive! Leads to extensivity.

Why is S^* not quite entropy?

Units! The logarithm is just a unitless quantity.

Thermal Equilibrium (Review)



$E_1 > E_2$

Isolated system that allows heat flow between 1 and 2.

$$\aleph(E_1, E - E_1) = \aleph_1(E_1) \bullet \aleph_2(E - E_1)$$

Number of micro states that give an particular energy distribution (E_1,E-E_1) is maximized with respect to E_1.

For a partitioning of E between 1 and 2, the number of accessible states is maximized when:

$$\left(\frac{\partial S_1^*}{\partial E_1}\right)_{N_1, V_1} = \left(\frac{\partial S_2^*}{\partial E_2}\right)_{N_2, V_2}$$

What do these partial derivatives relate to?

Thermal equilibrium --> Temperature!

$$\mathrm{d}E = T\mathrm{d}S\text{-}p\mathrm{d}V + \sum_{i=1}^{M} \mu_i \mathrm{d}N_i$$

Temperature $T = \left(\frac{\partial E}{\partial S}\right)_{V, N_i}$ or $\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{V, N_i}$

Summary

- Statistical Mechanics:
  - basic assumption:
    - all microstates are equally likely
  - Applied to NVE
    - Definition of Entropy: $S = k_B \ln \Omega$
    - Equilibrium: equal temperatures

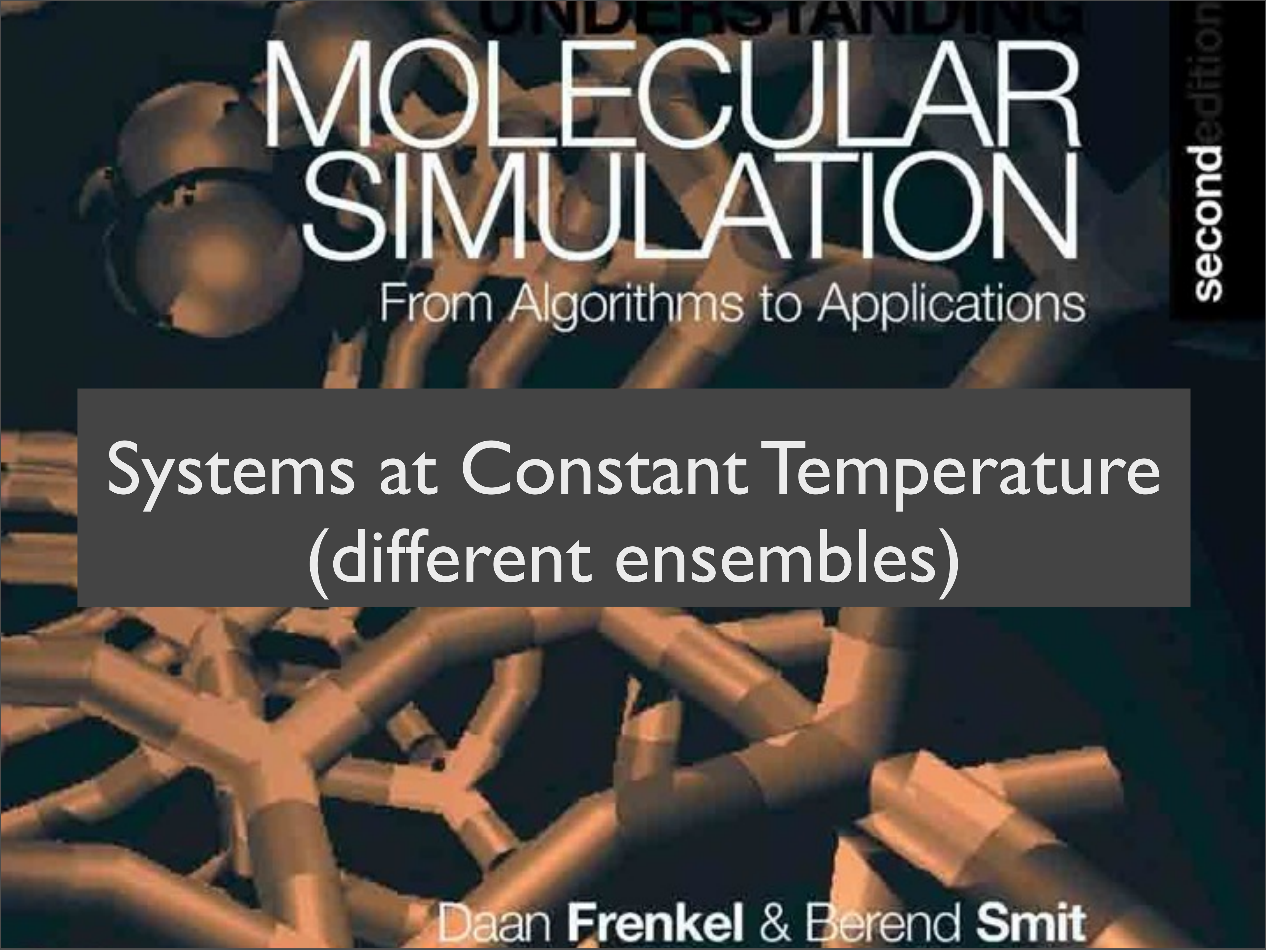
Question

How large is Ω for a glass of water?

- For macroscopic systems, super-astronomically large.
- For instance, for a glass of water at room temperature

$$\Omega \approx 10^{2\times10^{25}}$$

- Macroscopic deviations from the second law of thermodynamics are not forbidden, but they are extremely unlikely.



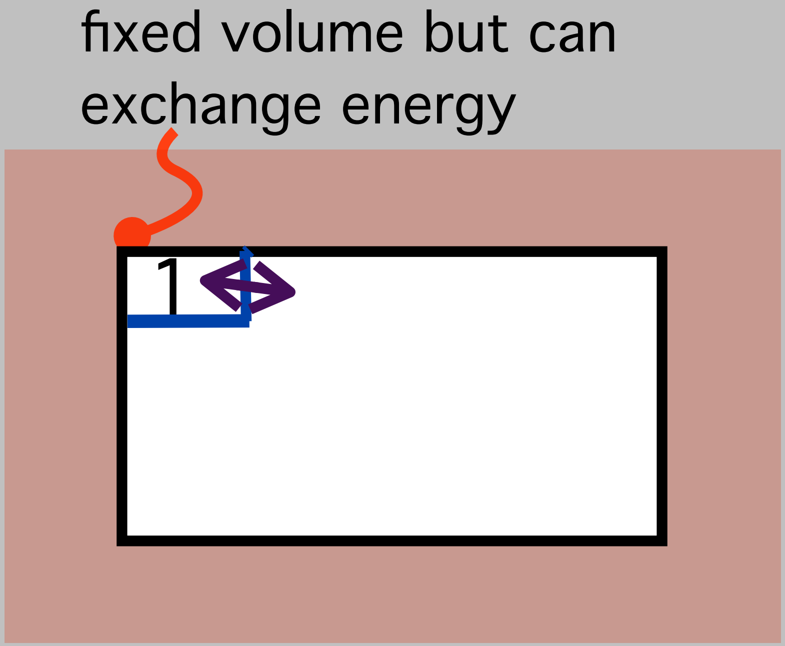
Systems at Constant Temperature (different ensembles)

The 2nd law

Entropy of an isolated system can only increase; until equilibrium were it takes its maximum value

Most systems are at constant temperature and volume or pressure?

What is the formulation for these systems?



Constant T and V

We have our box 1 and a bath

<u>Total system</u> is isolated and the volume is constant

First law $\mathrm{d}U = \mathrm{d}q - p\mathrm{d}V = 0$

Second law $\mathrm{d}S \geq 0$

<u>Box 1</u>: constant volume and temperature

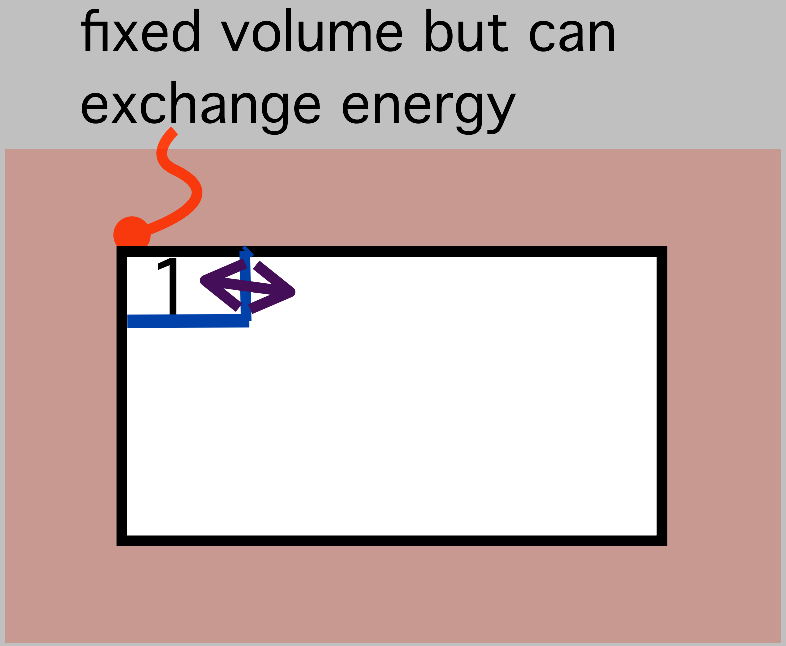
1st law: $\mathrm{d}U_1 + \mathrm{d}U_b = 0$ or $\mathrm{d}U_1 = -\mathrm{d}U_b$

The bath is so large that the heat flow does not influence the temperature of the bath + the process is reversible

2nd law: $\mathrm{d}S_1 + \mathrm{d}S_b = \mathrm{d}S_1 + \dfrac{\mathrm{d}U_b}{T} \geq 0$

$$T\mathrm{d}S_1 - \mathrm{d}U_1 \geq 0$$

Constant T and V



Total system is isolated and the volume is constant

Box 1: constant volume and temperature

2nd law: $T dS_1 - dU_1 \geq 0$

$$d(U_1 - TS_1) \leq 0$$

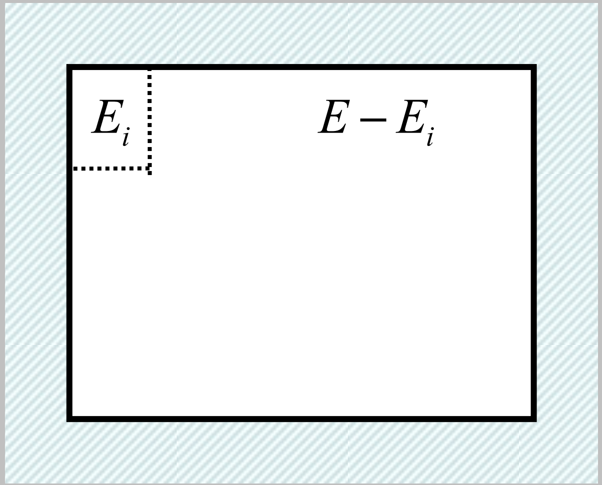
Let us define the Helmholtz free energy: A

$$A \equiv U - TS$$

For box 1 we can write $dA_1 \leq 0$

Hence, for a system at constant temperature and volume the Helmholtz free energy decreases and takes its minimum value at equilibrium

Canonical ensemble



Consider a small system that can exchange heat with a big reservoir

$$\ln \Omega(E - E_i) = \ln \Omega(E) - \frac{\partial \ln \Omega}{\partial E} E_i + \cdots$$

$1/k_B T$

$$\ln \frac{\Omega(E - E_i)}{\Omega(E)} = -\frac{E_i}{k_B T}$$

Hence, the probability to find E_i:

$$P(E_i) = \frac{\Omega(E - E_i)}{\sum_j \Omega(E - E_j)} = \frac{\exp(-E_i/k_B T)}{\sum_j \exp(-E_j/k_B T)}$$

$$P(E_i) \propto \exp(-E_i/k_B T)$$

Boltzmann distribution

Thermodynamics

What is the average energy of the system?

$$\langle E\rangle \equiv \sum_i E_i P(E_i) = \frac{\sum_i E_i \exp(-\beta E_i)}{\sum_j \exp(-\beta E_j)}$$

$$= -\frac{\partial \ln \sum_i \exp(-\beta E_i)}{\partial\beta}$$

$$= -\frac{\partial \ln Q_{N,V,T}}{\partial\beta}$$

Compare:

$$\left(\frac{\partial F/T}{\partial 1/T}\right) = E$$

Thermodynamics

First law of thermodynamics

$$\mathrm{d}E = T\mathrm{d}S - p\mathrm{d}V$$

Helmholtz Free energy:

$$F \equiv E - TS$$

$$\mathrm{d}F = -S\mathrm{d}T - p\mathrm{d}V$$

$$\left(\frac{\partial\, F/T}{\partial 1/T}\right) = F + \frac{1}{T}\frac{\partial F}{\partial 1/T} = F - T\frac{\partial F}{\partial T}$$

$$= F + TS = E$$

What is the average energy of the system?

$$\langle E \rangle \equiv \sum_i E_i P(E_i) = \frac{\sum_i E_i \exp(-\beta E_i)}{\sum_j \exp(-\beta E_j)}$$

$$= -\frac{\partial \ln \sum_i \exp(-\beta E_i)}{\partial \beta}$$

$$= -\frac{\partial \ln Q_{N,V,T}}{\partial \beta}$$

Compare:

$$\left(\frac{\partial F/T}{\partial 1/T}\right) = E$$

Hence:

$$\boxed{\frac{F}{k_B T} = -\ln Q_{N,V,T}}$$

Atoms?

We have assumed that we can count states

Quantum Mechanics: energy discreet

What to do for classical model such as an ideal gas, hard spheres, Lennard-Jones?

Energy is continue:

- potential energy
- kinetic energy

Particle in a box:

$$\varepsilon_n = \frac{(nh)^2}{8mL^2}$$

What are the energy levels for Argon in a 1-dimensional box of 1 cm?

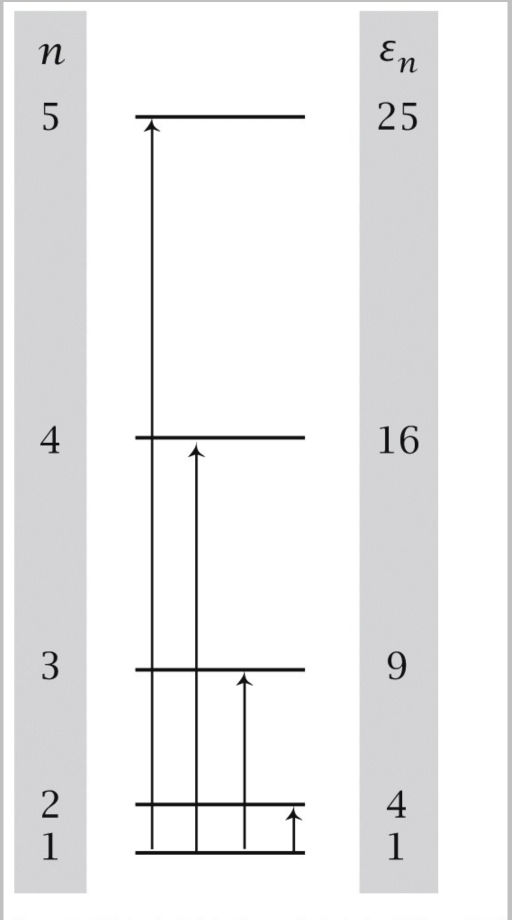


Figure 11.4 Molecular Driving Forces 2/e (© Garland Science 2011)

What are the energy levels for Argon in a 1-dimensional box of 1 cm?

$$\varepsilon_n = \frac{(nh)^2}{8mL^2}$$

Argon: m=40 g/mol=6.63×10^{-26} kg
h=6.63×10^{-34} J s

$$\varepsilon_n = 5\times10^{-39} n^2 (J)$$

Kinetic energy of Ar at room temperature ≈4.14 × 10^{-21} J

Many levels are occupied: only at very low temperatures or very small volumes one can see quantum effect!

$$q_{translational} = \sum_{n=1}^{\infty} e^{-\frac{(nh)^2}{8mL^2k_BT}}$$

$$q_{translational} = \int_0^{\infty} e^{-\frac{(nh)^2}{8mL^2k_BT}}\, dn$$

$$q_{translational} = \left(\frac{2\pi mk_BT}{h^2}\right)^{\frac{1}{2}} L$$

3D: $$q_{translational} = \left(\frac{2\pi mk_BT}{h^2}\right)^{\frac{3}{2}} V = \frac{V}{\Lambda^3}$$

De Broglie wavelength

Partition function; $q = \sum_{n=1}^{\infty} e^{-E_n} = \int e^{-E_n}\, dn$

Hamiltonian: $H = U_{kin} + U_{pot} = \sum_i \frac{p_i^2}{2m_i} + U_{pot}\left(r^N\right)$

$$Z_{1,V,T} = C\int e^{-\frac{p^2}{2m\, k_B T}}\, dp^{3N} \int e^{-\frac{U_p(r)}{k_B T}}\, dr^{3N}$$

One ideal gas molecule: $U_p(r)=0$

$$Z_{1,V,T}^{IG} = C\int e^{-\frac{p^2}{2m\, k_B T}}\, dp \int dr = CV\left(2\pi m k_B T\right)^{\frac{3}{2}}$$

$$q_{translational} = \left(\frac{2\pi m k_B T}{h^2}\right)^{\frac{3}{2}} V = \frac{V}{\Lambda^3}$$

$$Z_{1,V,T}^{IG} = CV\left(2\pi m k_B T\right)^{\frac{3}{2}} = \frac{V}{\Lambda^3}$$

N gas molecules:

$$Z_{N,V,T} = \frac{1}{h^{3N}} \int e^{-\frac{p^2}{2m\, k_B T}} dp^N \int e^{-\frac{U(r)}{k_B T}} dr^N$$

WRONG!
Particles are
indistinguishable

$$Z_{N,V,T} = \frac{1}{h^{3N} N!} \int e^{-\frac{p^2}{2m\, k_B T}} dp^N \int e^{-\frac{U(r)}{k_B T}} dr^N$$

Configurational part of
the partition function:

$$\boxed{Q_{N,V,T} = \frac{1}{\Lambda^{3N} N!} \int e^{-\frac{U(r)}{k_B T}} dr^N}$$

Question

- For an ideal gas, calculate:
  - the partition function
  - the pressure
  - the energy
  - the chemical potential

ideal gas molecules:

$$Q_{N,V,T}^{IG} = \frac{V^N}{\Lambda^{3N} N!}$$

All thermodynamics follows from the partition function!

Free energy:

$$F^{IG} = -k_B T \ln Q_{N,V,T}^{IG} = k_B T N \left[\ln \Lambda^3 - \ln\left(V/N \right) \right]$$

$$F^{IG} = F^0 + k_B T N \ln \rho$$

Pressure:

$$p = -\left(\frac{\partial F}{\partial V} \right)_{T,N} = k_B T N \frac{1}{V}$$

Energy:

$$E = \left(\frac{\partial F/T}{\partial 1/T} \right)_{V,N} = 3 k_B N \left(\frac{\partial \ln \Lambda}{\partial 1/T} \right)_{V,N}$$

$$E = \frac{3}{2} N k_B T$$

$$\Lambda = \left(\frac{h^2}{2\pi m k_B T} \right)^{\frac{1}{2}}$$

Chemical potential: $\mu_{\mathrm{i}} = \left(\dfrac{\partial \mathrm{F}}{\partial N_i} \right)_{T,V,N_j}$

$$\beta F = N \ln \Lambda^3 + N \ln\left(\frac{N}{V} \right)$$

$$\beta\mu = \ln \Lambda^3 + \ln \rho + 1$$

$$\beta\mu^{IG} = \beta\mu^0 + \ln \rho$$

Summary:
Canonical ensemble (*N*, *V*, *T*)

Partition function:

$$\boxed{Q(N,V,T)=\frac{1}{\Lambda^{3N}N!}\int d\mathbf{r}^N \exp\left[-\beta U\left(r^N\right)\right]}$$

Probability to find a particular configuration

$$\boxed{P(\Gamma)\propto \exp\left[-\beta U(\Gamma)\right]}$$

Free energy

$$\boxed{\beta F=-\ln Q_{N,V,T}}$$

Summary:
micro-canonical ensemble (*N,V,E*)

Partition function:

$$\boxed{Q(N,V,E)=\frac{1}{h^{3N}N!}\iint \mathrm{d}\mathbf{p}^N \mathrm{d}\mathbf{r}^N \delta\left(H\left(\mathbf{p}^N,\mathbf{r}^N\right)-E\right)}$$

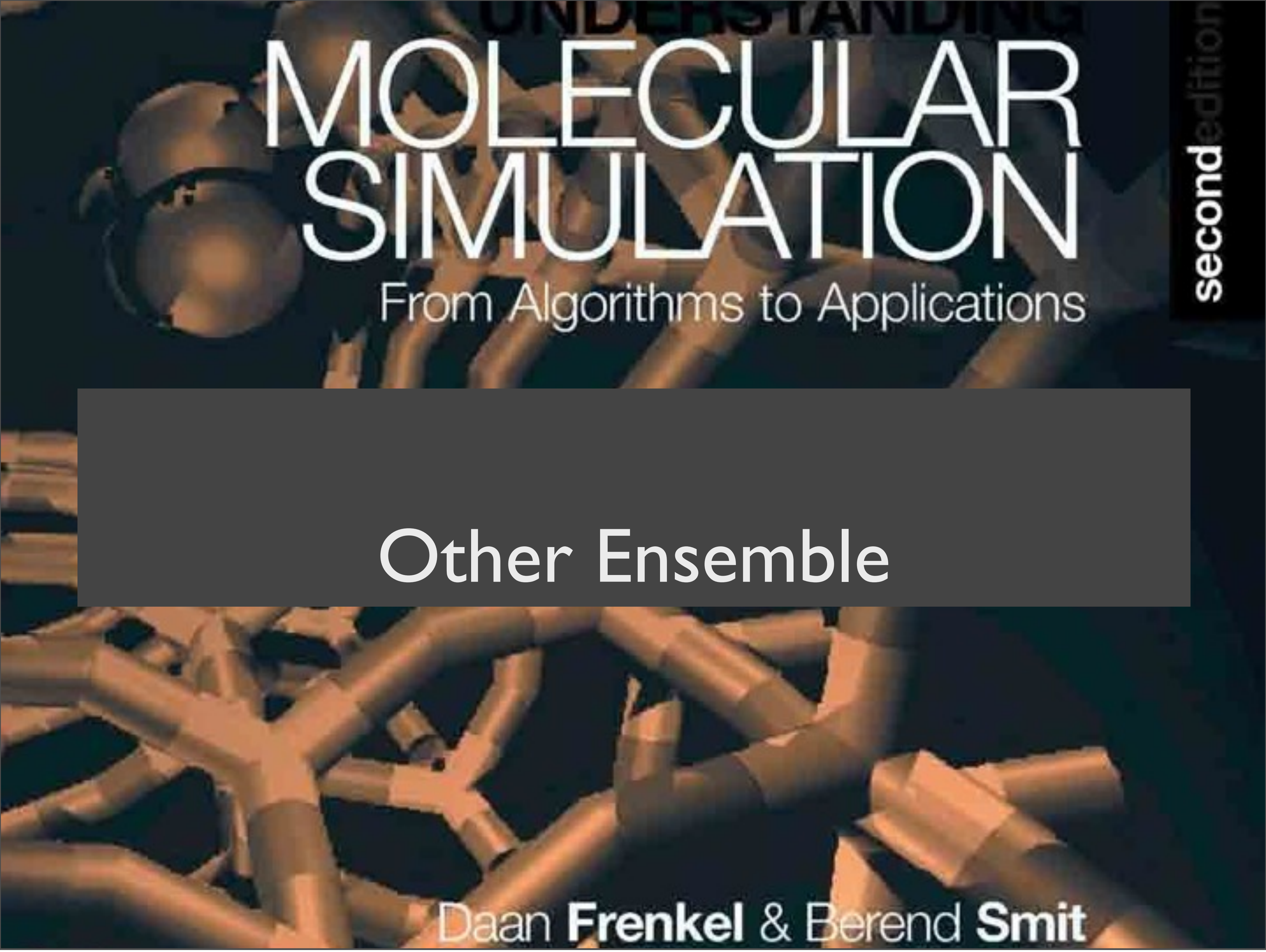
Probability to find a particular configuration

$$\boxed{P(\Gamma)\propto 1}$$

Free energy

$$\boxed{\beta S=\ln Q_{N,V,E}}$$

Other Ensemble



Other ensembles?

COURSE:
MD and MC different ensembles

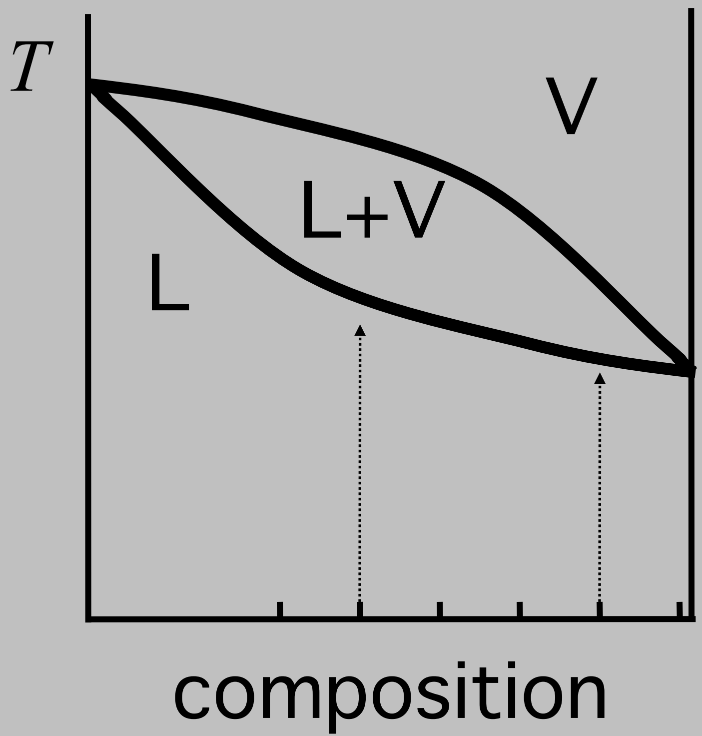
In the thermodynamic limit the thermodynami
independent of the ensemble: so buy a bigger

However, it is most of the times much better to th and to carefully select an appropriate ensemble.

For this it is important to know how to simulate in the various ensembles.

But for doing this wee need to know the Statistical Thermodynamics of the various ensembles.

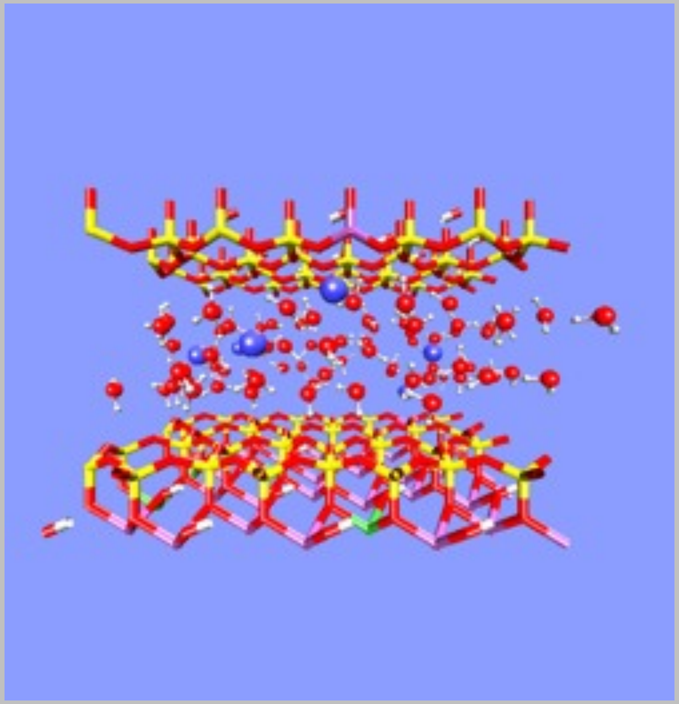
Example (1):
vapour-liquid equilibrium mixture



Measure the composition of the coexisting vapour and liquid phases if we start with a homogeneous liquid of two different compositions:

- How to mimic this with the N,V,T ensemble?
- What is a better ensemble?

Example (2): swelling of clays

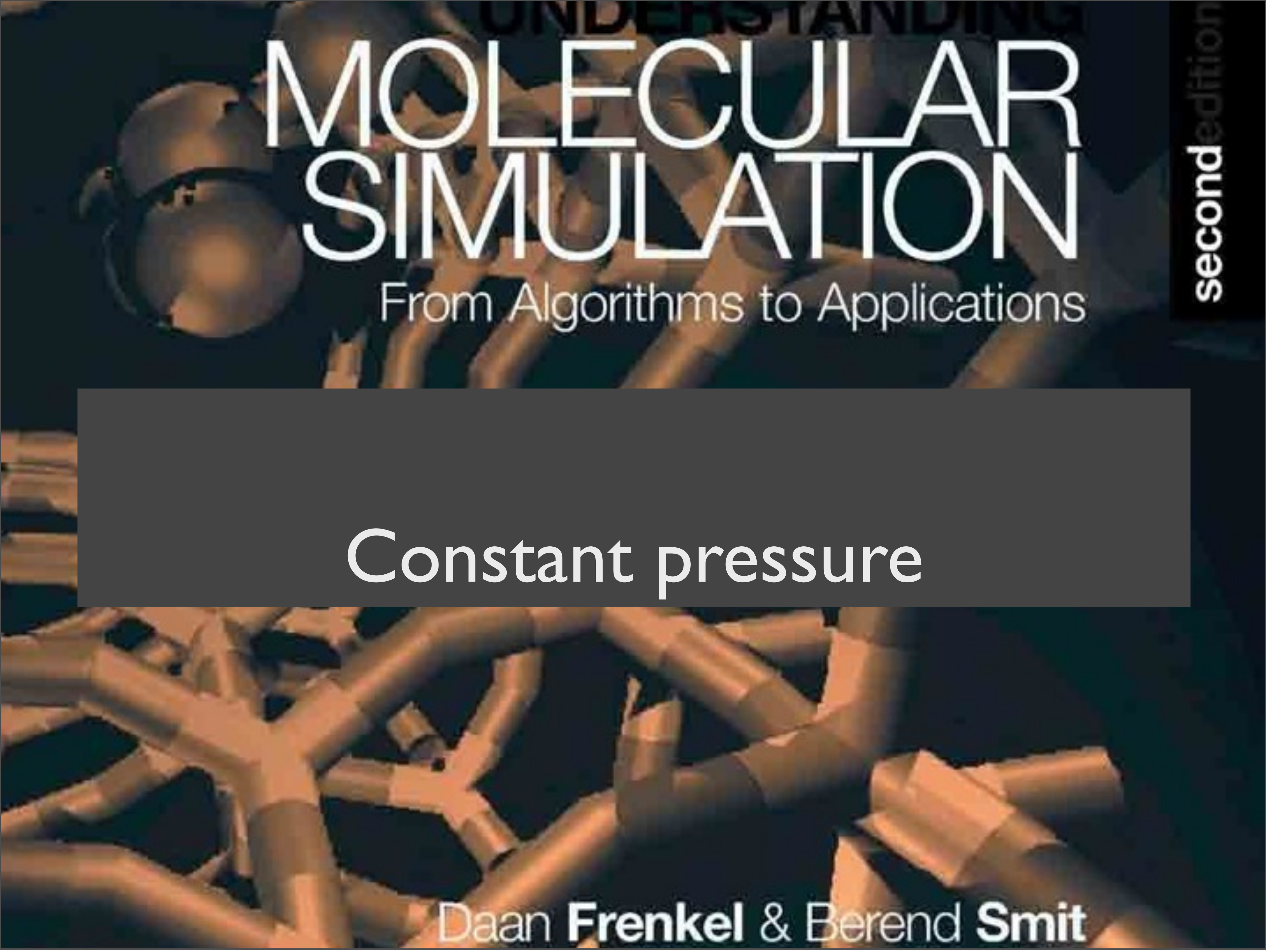


Deep in the earth clay layers can swell upon adsorption of water:

- How to mimic this in the N,V,T ensemble?
- What is a better ensemble to use?

Ensembles

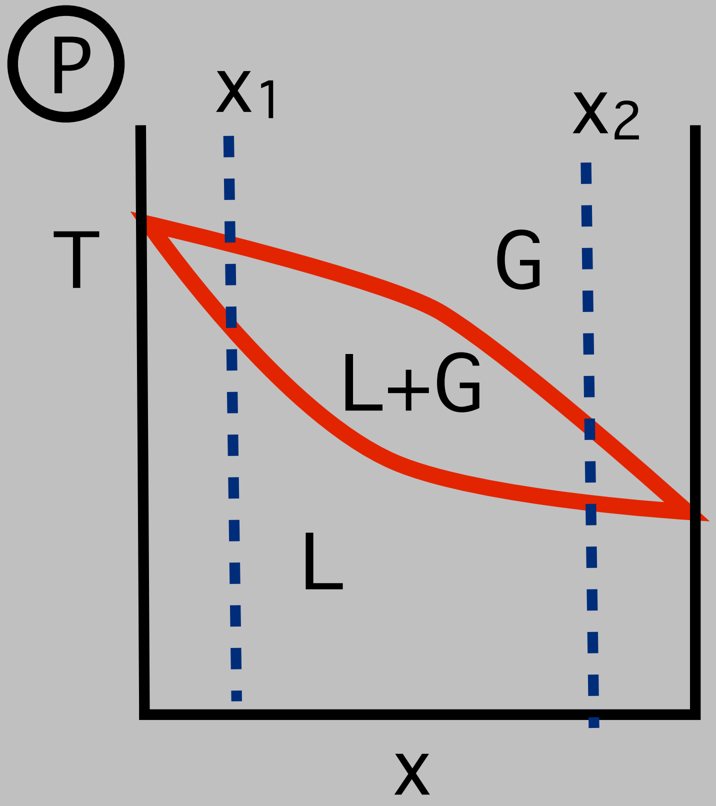
- Micro-canonical ensemble: E,V,N
- Canonical ensemble: T,V,N
- Constant pressure ensemble: T,P,N
- Grand-canonical ensemble: T,V,μ



Constant pressure

Experimental Example

Phase equilibria

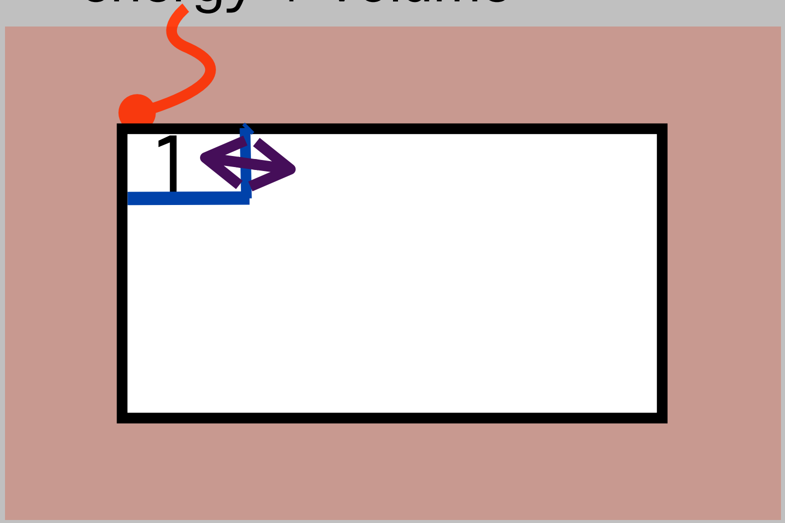


How do we measure vapor-liquid equilibria for a mixture?

How to mimic this experiment in NVT conditions?

Better solution: NPT ensemble

fixed N but can exchange energy + volume



We have our box 1 and a bath

Total system is isolated and the volume is constant

First law $dU = dq - pdV = 0$

Second law $dS \geq 0$

Box 1: constant pressure and temperature

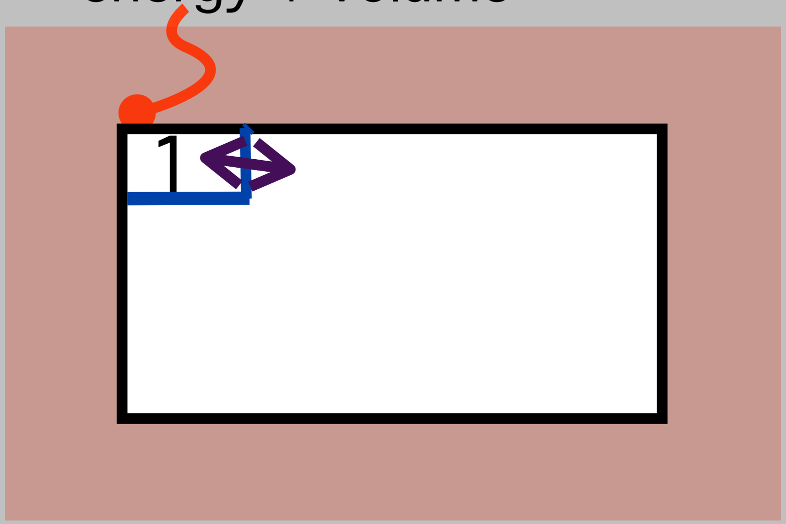
1st law: $dU_1 + dU_b = 0$ or $dU_1 = -dU_b$

$dV_1 + dV_b = 0$ or $dV_1 = -dV_b$

The bath is very large and the small changes do not change P or T; in addition the process is reversible

2nd law: $$dS_1 + dS_b = dS_1 + \frac{dU_b}{T} + \frac{p}{T}dV_b \geq 0$$

$$TdS_1 - dU_1 - pdV_1 \geq 0$$



Total system is isolated and the volume is constant

Box 1: constant pressure and temperature

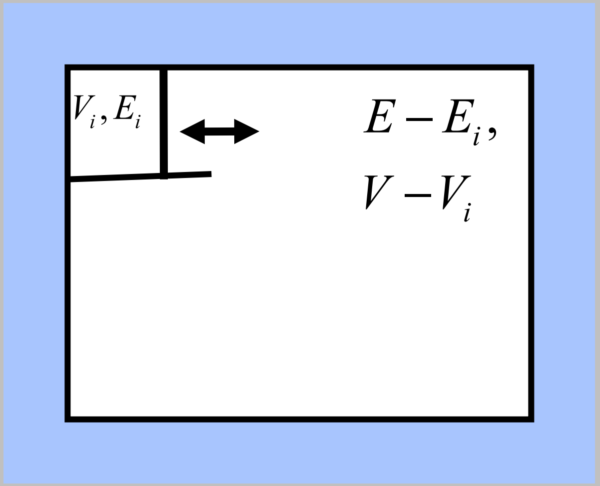
2nd law: $TdS_1 - dU_1 - pdV_1 \geq 0$

$$d(U_1 - TS_1 + pV_1) \leq 0$$

Let us define the **Gibbs free energy**: G $\quad G \equiv U - TS + pV$

For box 1 we can write $dG_1 \leq 0$

Hence, for a system at constant temperature and pressure the Gibbs free energy decreases and takes its minimum value at equilibrium



N,P,T ensemble

Consider a small system that can exchange volume and energy with a big reservoir

$$\ln\Omega\left(V-V_{i,}E-E_i\right)=\ln\Omega\left(V,E\right)-\left(\frac{\partial\ln\Omega}{\partial E}\right)_V E_i-\left(\frac{\partial\ln\Omega}{\partial V}\right)_E V_i+\cdots$$

The terms in the expansion follow from the connection with Thermodynamics:

$$S=k_B\ln\Omega$$

We have:

$$dS=\frac{1}{T}dU+\frac{p}{T}dV-\sum\frac{\mu_i}{T}dN_i$$

$$\left(\frac{\partial S}{\partial U}\right)_{V,N_i}=\frac{1}{T}$$

and

$$\left(\frac{\partial S}{\partial V}\right)_{E,N_i}=\frac{p}{T}$$

$$\ln\Omega\left(V-V_i,E-E_i\right)=\ln\Omega\left(V,E\right)-\left(\frac{\partial\ln\Omega}{\partial E}\right)_{V,N}E_i-\left(\frac{\partial\ln\Omega}{\partial V}\right)_{E,N}V_i+\cdots$$

$$\ln\Omega\left(V-V_i,E-E_i\right)=\ln\Omega\left(V,E\right)-\frac{E_i}{k_BT}-\frac{p}{k_BT}V_i$$

$$\ln\frac{\Omega\left(E-E_i,V-V_i\right)}{\Omega\left(E,V\right)}=-\frac{E_i}{k_BT}-\frac{pV_i}{k_BT}$$

Hence, the probability to find E_i,V_i:

$$P\left(E_i,V_i\right)=\frac{\Omega\left(E-E_i,V-V_i\right)}{\sum_{j,k}\Omega\left(E-E_j,V-V_k\right)}=\frac{\exp\left[-\beta\left(E_i+pV_i\right)\right]}{\sum_{j,k}\exp\left[-\beta\left(E_j+pV_k\right)\right]}$$

$$\propto\exp\left[-\beta\left(E_i+pV_i\right)\right]$$

Partition function: $$\Delta(N,P,T)=\sum_{i,j}\exp\left[-\frac{E_i}{k_BT}-\frac{pV_j}{k_BT}\right]$$

Ensemble average:

$$\langle V\rangle=\frac{\sum_{i,j}V_j\exp\left[-\frac{E_i}{k_BT}-\frac{pV_j}{k_BT}\right]}{\Delta(N,p,T)}=-k_BT\left(\frac{\partial\ln\Delta}{\partial p}\right)_{T,N}$$

Thermodynamics $$dG=-SdT+Vdp+\sum\mu_i dN_i$$

$$V=\left(\frac{\partial G}{\partial p}\right)_{T,N}$$

Hence:

$$\boxed{\frac{G}{k_BT}=-\ln\Delta(N,p,T)}$$

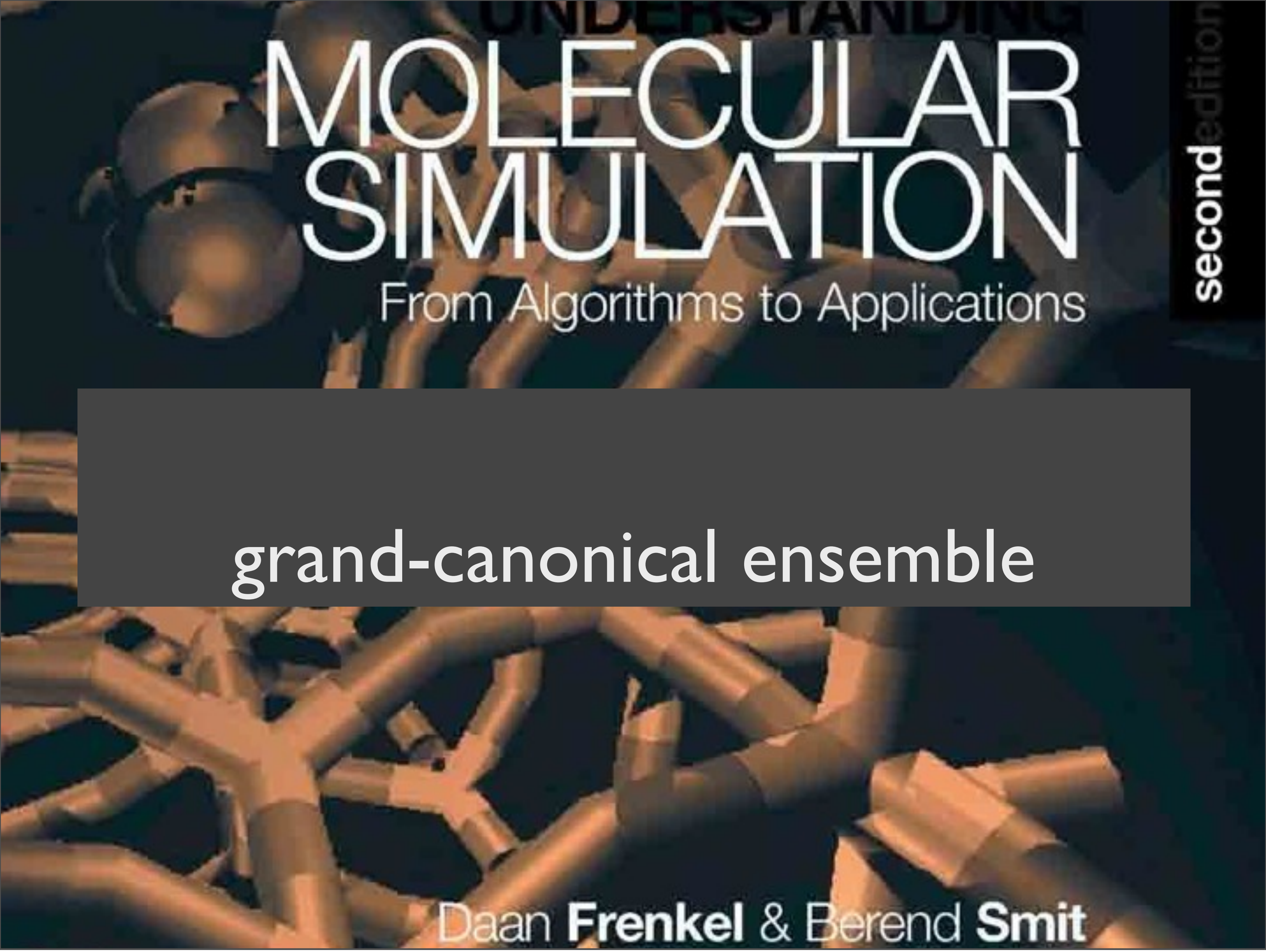
Summary

In the classical limit, the partition function becomes

$$Q(N,P,T)=\frac{1}{\Lambda^{3N}N!}\int dV\exp(-\beta PV)\int \mathrm{d}\mathbf{r}^N\exp\left[-\beta U\left(r^N\right)\right]$$

The probability to find a particular configuration: $\mathbf{r}^N,V$

$$P\left(\mathbf{r}^N,V\right)\propto\exp\left[-\beta\left(PV+U\left(r^N\right)\right)\right]$$



grand-canonical ensemble

Grand-canonical ensemble

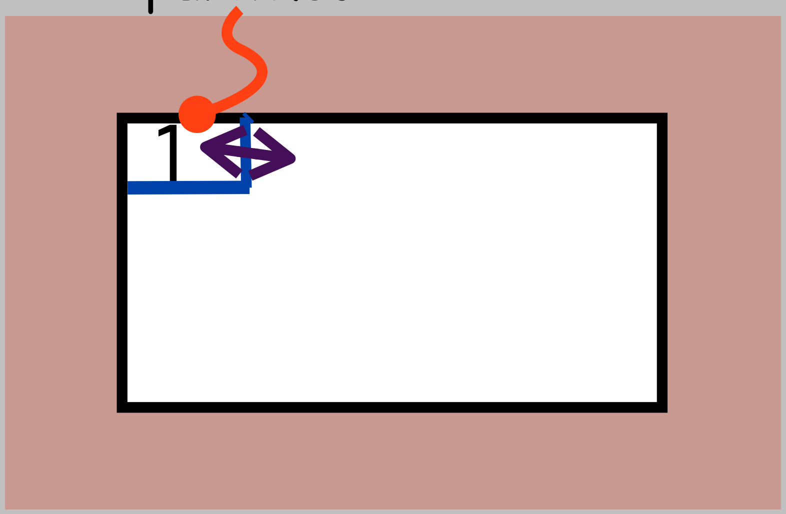
Classical

- A small system that can exchange heat and particles with a large bath

Statistical

- Taylor expansion of a small reservoir

Constant T and μ



Total system is isolated and the volume is constant

First law

$$dU = TdS - pdV + \mu dN = 0$$

Second law $dS \geq 0$

Box 1: constant chemical potential and temperature

1st law: $dU_1 + dU_b = 0$ or $dU_1 = -dU_b$

$dN_1 + dN_b = 0$ or $dN_b = -dN_1$

The bath is very large and the small changes do not change μ or T; in addition the process is reversible

2nd law:

$$dS_1 + dS_b = dS_1 + \frac{1}{T_b} dU_b - \frac{\mu_b}{T_b} dN_b \geq 0$$

$$dS_1 + dS_b = dS_1 + \frac{1}{T_b} dU_b - \frac{\mu_b}{T_b} dN_b \geq 0$$

We can express the changes of the bath in terms of properties of the system

$$dS_1 - \frac{1}{T} dU_1 + \frac{\mu}{T} dN_1 \geq 0 \qquad d\left(TS_1 - U_1 + \mu N_1\right) \geq 0$$

$$d\left(U - TS - \mu N\right) \leq 0$$

For the Gibbs free energy we can write:

$$G \equiv U - TS + pV$$

or

$$-pV = U - TS - \mu N$$

$$G = \mu N$$

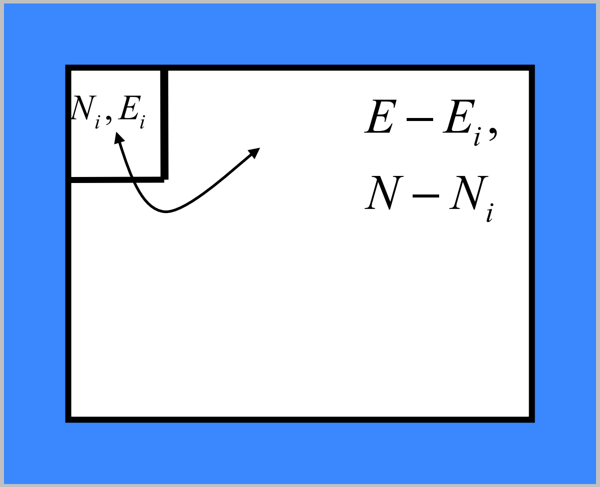
Giving:

$$d\left(-pV\right) \leq 0$$

or

$$d\left(pV\right) \geq 0$$

Hence, for a system at constant temperature and chemical potential pV increases and takes its maximum value at equilibrium



μ, V, T ensemble

Consider a small system that can exchange particles and energy with a big reservoir

$$\ln\Omega\left(E - E_i, N - N_j,\right) = \ln\Omega\left(E, N\right) - \left(\frac{\partial \ln\Omega}{\partial E}\right)_{V,N} E_i - \left(\frac{\partial \ln\Omega}{\partial N}\right)_{E,V} N_j + \cdots$$

The terms in the expansion follow from the connection with Thermodynamics:

$$S = k_B \ln\Omega$$

$$dS = \frac{1}{T} dU + \frac{p}{T} dV - \frac{\mu}{T} dN$$

$$\left(\frac{\partial S}{\partial U}\right)_{V,N} = \frac{1}{T} \qquad \left(\frac{\partial S}{\partial N}\right)_{E,V} = -\frac{\mu}{T}$$

$$\ln\Omega\left(E-E_i,N-N_j,\right)=\ln\Omega\left(E,N\right)-\left(\frac{\partial\ln\Omega}{\partial E}\right)_{V,N}E_i-\left(\frac{\partial\ln\Omega}{\partial N}\right)_{E,V}N_j+\cdots$$

$$\ln\Omega\left(E-E_i,N-N_j\right)=\ln\Omega\left(E,N\right)-\frac{E_i}{k_BT}+\frac{\mu}{k_BT}N_j$$

$$\ln\frac{\Omega\left(E-E_i,N-N_j\right)}{\Omega\left(E,N\right)}=-\frac{E_i}{k_BT}+\frac{\mu}{k_BT}N_j$$

Hence, the probability to find E_i,N_j:

$$P\left(E_i,N_j\right)=\frac{\Omega\left(E-E_i,N-N_j\right)}{\sum_{k,l}\Omega\left(E-E_k,N-N_l\right)}\propto\exp\left[-\frac{E_i}{k_BT}+\frac{\mu N_i}{k_BT}\right]$$

μ,V,T ensemble (2)

In the classical limit, the partition function becomes

$$Q(\mu,V,T)=\sum_{N=1}^{\infty}\frac{\exp(\beta\mu N)}{\Lambda^{3N}N!}\int d\mathbf{r}^N \exp\left[-\beta U\left(r^N\right)\right]$$

The probability to find a particular configuration:

$$N,\mathbf{r}^N$$

$$P\left(N,\mathbf{r}^N\right)\propto \exp\left[\beta\mu N-\beta U\left(r^N\right)\right]$$