A brief introduction to Green's functions

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Prologue

Greens's functions allow to find solutions for linear differential equations

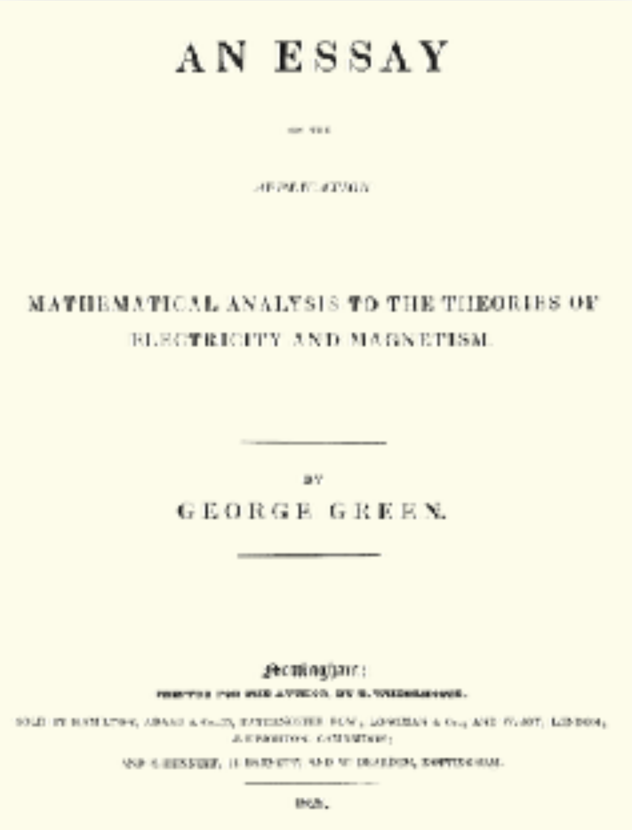
- ordinary (ODE) or partial (PDE)
- inhomogeneous

provided that there are enough (homogeneous or not) initial and boundary conditions.

The Green's function is defined by a similar differential equation,
in which the inhomogeneous term is a delta function.

George Green

George Green (14 July 1793 – 31 May 1841) was a British mathematical physicist who wrote An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism (Green, 1828)



AN ESSAY

MATHEMATICAL ANALYSIS TO THE THEORIES OF
ELECTRICITY AND MAGNETISM.

BY
GEORGE GREEN.

Not George Green.
Well, the name of this man is actually George Green. According to the Italian Wikipedia, this is even our mathematician; but this is incorrect. This George Green was a shipbuilder from Poplar, UK…

Differential equations

In physics, differential equations are everywhere

Laplace equation

$$\nabla^2 \phi = 0$$

second-order
(elliptical) PDE

Maxwell-Faraday equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

classical harmonic oscillator

$$m \frac{d^2 x}{dt^2} = -kx$$

second-order ODE

time-dependent Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}, t) = \left[\frac{-\hbar^2}{2\mu} \nabla^2 + V(\mathbf{r}, t) \right] \Psi(\mathbf{r}, t)$$

linear PDE

heat equation (type of diffusion equation)
(e.g. Fick’s law)

$$\frac{\partial \phi(\mathbf{r}, t)}{\partial t} = D \nabla^2 \phi(\mathbf{r}, t)$$

parabolic PDE

Langevin equation

$$m \frac{d^2 \mathbf{r}}{dt^2} = -\lambda \frac{d\mathbf{x}}{dt} + \eta(t)$$

Poisson equation

$$\nabla^2 \phi = -\frac{\rho}{\epsilon}$$

second-order
(elliptical) PDE

Liouville’s equation

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2} \right] \log f = -K f^2$$

nonlinear PDE

Differential equations

In physics, differential equations are everywhere

Note, many equations represent:

$\mathcal{L}$(response) = source

differential operator

for example:

$\mathcal{L}$(electric field) = charge distribution

causes

Laplace equ

$\nabla^2\phi =$

second-or
(elliptical) P

tim

$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},$

usion equation)
w)

$\phi(\mathbf{r}, t)$

Langevin

$$m\frac{d^2\mathbf{r}}{dt^2} = -\lambda\frac{d\mathbf{r}}{dt} + \eta(t)$$

ϵ

$= -Kf^2$

second-order
(elliptical) PDE

nonlinear PDE

The differential operator

Consider the following homogeneous ordinary differential equation (ODE):

$$\frac{d^2 f(x)}{dx^2} + x^2 f(x) = 0 \quad \text{=>} \quad \left(\frac{d^2}{dx^2} + x^2\right) f(x) = 0$$

$$\mathcal{L} f(x) = 0$$

differential operator

$$\mathcal{L} = \frac{d^2}{dx^2} + x^2$$

(linear combination of derivative operators times functions)

- The inverse of a differential operator is an integral operator
- G is kernel of the integral operator

Here, the Green's function is the inverse of the differential operator:

$$G(x,y) \sim \mathcal{L}^{-1} \sim \left(\frac{d^2}{dx^2} + x^2\right)^{-1}$$

note: G has 2 variables (explained hereafter…)

Consider now the inhomogeneous problem: $\mathcal{L} f(x) = g(x)$

which can now be solved by the analogue of $f(x) = G(x,y)g(x)$

$$\mathcal{L} f(x) \sim \mathcal{L} G(x,y) g(x)$$

$$\mathcal{L}\mathcal{L}^{-1} g(x) = g(x)$$

General idea: Green's function is the inverse of an arbitrary linear differential operator

Dirac delta function

The Green's function is the solution with a <u>point (or impulse) source</u> as the forcing function

$$\mathcal{L}G(x,y) = \delta(x-y)$$

Dirac delta function
- δ(x-y) = 0 if x≠y
- $\int dx\, \delta(x-x') = 1$

inhomogeneous DE

$$\mathcal{L}f(x) = \underline{g(x)}$$

aka forcing function, or driving term

- technically, the delta function is not a function but a distribution
- also the Green's function can be seen as a distribution

Now, the solution to an arbitrary non-homogeneous forcing term, $\mathcal{L}u(x) = f(x)$, is:

$$u(x) = \int G(x,y)f(y)dy$$

since: $\mathcal{L}u(x) = \int \mathcal{L}G(x,y)f(y)dy = \int \delta(x-y)f(y)dy = f(x)$

adding an uncountable superposition of solutions with point source to the arbitrary forcing term

So, the solution can be built up point by point by integrating the Green's function against the forcing term.

The resulting integral may be difficult or impossible to solve, but can be computer numerically.

Linear differential operators

The green's function approach works for linear differential equations

Linearity allows for superposition:

$$\mathcal{L}(f + g) = \mathcal{L}f + \mathcal{L}g$$

Suppose that we know the answer for a homogenous problem.

$$\mathcal{L}f = 0$$

Then, for the case of some inhomogeneity g, guess a solution that works for this driving term.

$$\mathcal{L}f = g$$

Adding homogeneous solution still solves the original equation, because we are just adding zero on both sides!

Electrostatic potential

The problem of finding the potential due to a given charge distribution

Consider the Poisson equation: $\nabla^2 \phi = -\frac{\rho}{\epsilon_0}$

charge distribution

electric potential

Boundary condition:
ϕ = 0 at r = ∞

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$$

Has well-known Green's function: $G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}'|}$

Coulomb potential
of q=1 unit charge

The point source is here a point charge at **r** = **r**'

Electrostatic potential

Importance of knowing this Green's function solution:

$$\mathcal{L}G(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')$$

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}'|}$$

multiply both side by *f*(**r'**) and integrate over **r'**

$$\mathcal{L}\int G(\mathbf{r},\mathbf{r}')f(\mathbf{r}')d\mathbf{r}' = \int \delta(\mathbf{r}-\mathbf{r}')f(\mathbf{r}')d\mathbf{r}' = f(\mathbf{r})$$

- f(**r'**) is a constant
- **r'** is the auxiliary variable
- ***ℒ*** independent of **r'**

So, adding up a set of impulses on the right, and sifting with the delta function, gives *f*(**r**).

General solution: a superposition of different Green's functions

$$y(\mathbf{r}) = \int G(\mathbf{r},\mathbf{r}')f(\mathbf{r}')d\mathbf{r}'$$

continuous charge distribution:

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r})}{|\mathbf{r}-\mathbf{r}'|}dV'$$

collection of point charges q_k:

$$\phi = \frac{1}{4\pi\epsilon_0}\sum_k \frac{q_k}{r_k}$$

the GF is a propagator
it propagates the
response of an impulse at
r' to some other point **r**

Electrostatic potential

In sum, to solve by Green’s function a Poisson equation: $\nabla^2\phi = -\rho(\mathbf{r})$

1. consider simpler problem using delta function

$$\nabla^2\phi = -\delta(\mathbf{r}-\mathbf{r}')$$

2. find its solution; this is the Green’s function

$$\phi_{\mathbf{r}'}(\mathbf{r}) = G(\mathbf{r},\mathbf{r}')$$

3. write the solution to the original problem as an integral

$$\phi(\mathbf{r}) = \int d^3\mathbf{r}'\, G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}') = \int d^3\mathbf{r}'\, \frac{\rho(\mathbf{r}')}{4\pi\epsilon|\mathbf{r}-\mathbf{r}'|}$$

- Finding the Green’s function is often relatively easy
- Many different problems have the same Green’s functions

Harmonic oscillator

Classical harmonic oscillator equation:

$$m\ddot{x} + kx = 0$$

Solution:

$$x = A\sin(\omega t) + B\cos(\omega t) \qquad \omega = \sqrt{\frac{k}{m}}$$

A and B are arbitrary constants
determined by boundary conditions (initial position and velocity)

Forced harmonic oscillator equation:

$$m\ddot{x} + kx = F(t)$$

How to find a general solution for any arbitrary forcing function? Hopeless?

Linearity and superposition (again)

If we have solutions for two different forcing functions:

$$m\ddot{x}_1 + kx + 1 = F_1(t)$$

$$m\ddot{x}_2 + kx + 2 = F_2(t)$$

Then we also know the solution for forcing function $F_1 + F_2$:

$$m\frac{d^2}{dt^2}\big(x_1(t) + x_2(t)\big) + k\big(x_1(t) + x_2(t)\big) = F_1(t) + F_2(t)$$

Suggestion:

1. choose simple set of forcing functions
2. solve for these simple functions
3. for arbitrary *F(t)*, write solution as linear combination of simple solutions

George Green: use delta functions!

Harmonic oscillator

To solve for impulse force at one point, *t'*, in time

$$m\ddot{x} + kx = \delta(t - t')$$

requires initial conditions.

let's take: x(t) = 0 for t < 0

- initially oscillator at rest (we can later take other conditions, thanks to linearity)
- this gives: *retarded Green's function*, effects appear after force is applied

for t > 0

- force is again zero
- solution of form: $x = A\sin(\omega t) + B\cos(\omega t)$

Now, connect solution for t < 0 with that of t > 0

integrate both sides over small interval:

$$\int_{-\epsilon}^{\epsilon} m\ddot{x}(t)\,dt + \int_{-\epsilon}^{\epsilon} kx(t)\,dt = \int_{-\epsilon}^{\epsilon} \delta(t)\,dt$$

$x(t = 0^+) = x(t = 0^-) = 0$ => B=0

$\dot{x}(t = 0^+) = 1/m$ => Aω=1/m

gives:

$$x(t) = \frac{1}{m\omega}\sin(\omega t) \qquad t > 0$$

our Green's function

In sum, an impulse at t=0 on oscillator at rest, converts oscillator in motion

Harmonic oscillator

Next: an arbitrary forcing function: $m\ddot{x} + kx = F(t)$

Build the forcing function from delta functions:

$$F(t) = \int_{-\infty}^{\infty} F(t')\delta(t-t')\,dt'$$

Green’s function solution:

$$x(t) = \int_{-\infty}^{t} F(t')\frac{1}{m\omega}\sin(\omega(t-t'))\,dt'$$

Causality:
to find x(t), add up all delta functions at *t’*< *t*, but not those at *t’*> *t*

(homework: check that this is indeed a solution for the above differential equation)

Scattering theory

Green functions can describe (nonrelativistic) scattering of a single particle by an external potential, *V(r)*

time-independent Schrödinger equation

particle of mass *m* and energy *E:*

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(\mathbf{r})\psi(\mathbf{r}) = E\,\psi(\mathbf{r})$$

assumption: wave packet has well-defined energy and is thus many wavelengths long. So it looks like a plane wave, independent of time during scattering event.

=> $$\left(\frac{\hbar^2}{2m}\nabla^2\psi + E_k\right)\psi(\mathbf{r}) = V(\mathbf{r})\,\psi(\mathbf{r})$$

$$\psi_k(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} + f(\theta,\phi)\frac{e^{ikr}}{r}$$

Treat potential as the charge density in the Poisson problem. Solution:

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + \int G(\mathbf{r},\mathbf{r}')\frac{2m}{\hbar^2}V(\mathbf{r}')\psi(\mathbf{r}')d\tau'$$

$\psi_0(\mathbf{r})$: incident wave

wave amplitude at **r** due to point source at **r’**

Problem: the solution contains the unknown $\psi(\mathbf{r})$, so iterative method needed.

Summary

- Ordinary and partial differential equations are used everywhere in the physical sciences. Often, they can be interpreted as a source inducing a response.
- The method of Green's functions is a powerful method to find solutions to certain linear differential equations.
- The Green's function is found as the impulse function using a Dirac delta function as a point source or force term.
- Solutions to the inhomogeneous ODE or PDE are found as integrals over the Green's function.
- Green's functions can be used to solve for example the Poisson equation in electrostatic problems, the driven or dampled harmonic oscillator, and the Schrödinger equation for scattering problems.