I Special Relativity

A. Classical Mechanics

Classical mechanics is based on the following facts / assumptions

i) Euclidean geometry applies
ii) Inertial frames exist
iii) Measurements in different inertial frames are related by Galilei transformations
iv) The equations of motion are invariant under Galilei transformations
v) When using Cartesian coordinates the equations of motion read

$$m \frac{d^2 x^a}{dt^2} = F^a$$

Remarks:

i) This implies that we can choose a frame of reference and use concepts like

parallel and perpendicular lines
distance between points
Cartesian coordinates

ii) A frame of reference with respect to which a free particle moves along straight lines is called inertial.
A frame of reference that moves with constant velocity along a straight line with respect to an inertial frame is itself an inertial frame.

iii) Using Cartesian coordinates $(x^a) = (x^1, x^2, x^3)$ in one inertial frame and similar coordinates $(\bar{x}^a)$ in another inertial frame that moves with velocities $(u^a) = (u^1, u^2, u^3)$ along the three axes of the first frame, the Galilei transformations read

$$\bar{t} = t$$
$$\bar{x}^a = x^a - u^a t$$

iv) This holds for mechanics, but not for all of physics. Maxwell's equations are not invariant under Galilei transformations. For the latter Lorentz found the transformation laws, which now carry his name.

v) This is Newton's equation of motion in Cartesean coordinates. When using other coordinates it looks quite a bit more complicated

B. Special Relativity

In order to turn classical mechanics into special relativity we must replace Galilei transformations by Lorentz transformations. We then get

i) Euclidean geometry applies
ii) Inertial frames exist
iii) Measurements in different inertial frames are related by Lorentz transformations
iv) Physics is invariant under Lorentz transformations
v) When using Cartesian coordinates the equations of motion read

$$m \frac{d^2 x^\alpha}{d\tau^2} = F^\alpha$$

Remarks:

i) No comment
ii) No comment
iii) Einstein derived the Lorentz transformations arguing

a/ Using Cartesian coordinates the transformations must be linear.
This is true because rectilinear motion is described by a linear equation and rectilinear motion in one inertial frame implies rectilinear motion in another inertial frame; the transformation should therefore transform a linear equation onto a linear equation.

b/ The speed of light has the same value in all inertial frames, irrespective of their relative motion or of the motion of the light source

c/ When using Cartesian coordinates and assuming frame B has velocities (u^a) with respect to frame A, then frame A has velocities $(-u^a)$ with respect to frame B.

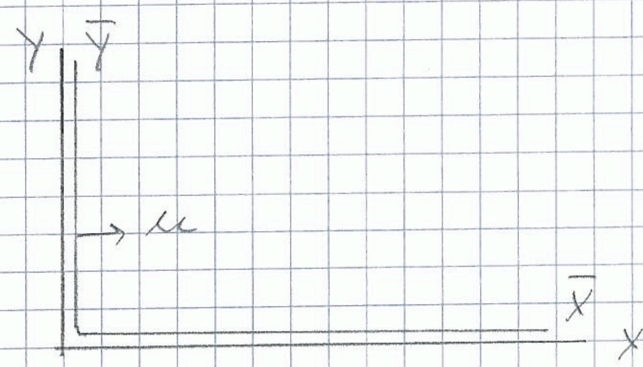
In case frame B with Cartesian coordinates $(\bar{x}^a)$ and time $\bar{t}$ moves with respect to frame A with Cartesian coordinates (x^a) and time t along their common X-axis, the transformations read:

$$c\bar{t} = \frac{1}{\sqrt{1-u^2/c^2}}\left(ct - \frac{u}{c}x\right)$$

$$\bar{x} = \frac{1}{\sqrt{1-u^2/c^2}}\left(-\frac{u}{c}ct + x\right)$$

$$\bar{y} = y$$

$$\bar{z} = z$$



By performing appropriate rotations this can be transformed into the transformations between inertial frames moving in any relative direction.

In general the Lorentz transformation leaves

$$c^2(t-t_e) - (\vec{r}-\vec{r}_e)\cdot(\vec{r}-\vec{r}_e) = 0$$

invariant (the speed of light is the same in all frames).
It is not difficult to check that the Lorentz transformation leaves the left-hand side invariant, which is a stronger result.
So, whichever coordinates are being used

$$c^2dt^2 - d\vec{r}\cdot d\vec{r}$$

has always the same value

Notation

$$(x^\alpha) = (ct, x^1, x^2, x^3)$$

$$\begin{aligned} ds^2 &= c^2 dt^2 - d\vec{r}\cdot d\vec{r} \\ &= c^2 dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \\ &= \sum_\alpha \sum_\beta \eta_{\alpha\beta}\, dx^\alpha dx^\beta \end{aligned}$$

$$(\eta_{\alpha\beta}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Einstein's summation convention

$$ds^2 = \eta_{\alpha\beta}\, dx^\alpha dx^\beta$$

Invariance means

$$\eta_{\alpha\beta}\, d\bar{x}^\alpha d\bar{x}^\beta = \eta_{\alpha\beta}\, dx^\alpha dx^\beta$$

ds^2 is called the line element

Generally

$$\bar{x}^\alpha = \Lambda^\alpha{}_\beta x^\beta$$

$$\eta_{\alpha\beta} d\bar{x}^\alpha d\bar{x}^\beta = \eta_{\alpha\beta} \Lambda^\alpha{}_\mu dx^\mu \Lambda^\beta{}_\nu dx^\nu = \eta_{\mu\nu} dx^\mu dx^\nu$$

$$\longrightarrow \quad \eta_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu = \eta_{\mu\nu}$$

iv) Physics is the same in all inertial frames

v) We must find an equation of motion that is invariant for Lorentz transformations, and becomes Newton's equation for velocities $\ll c$.

Consider the path of a moving particle and take two nearby points on this path:

$$ds^2 = c^2 dt^2 - d\vec{r}\cdot d\vec{r} = c^2 dt^2 - v^2 dt^2 = c^2 dt^2 (1 - v^2/c^2) := c^2 d\tau^2$$

$$d\tau = dt\sqrt{1 - v^2/c^2}$$

- for small velocities $d\tau \simeq dt$
- the numerical value of $d\tau$ is independent of the reference frame (if inertial)

- The physical meaning of $d\tau$ is the time span measured in a reference frame that happens to move with the particle; $d\tau$ is the eigentime of the particle.

Now consider

$$v^\alpha = \frac{dx^\alpha}{d\tau} \quad \rightarrow \quad \bar{v}^\alpha = \frac{d\bar{x}^\alpha}{d\tau} = \Lambda^\alpha{}_\beta \frac{dx^\beta}{d\tau} = \Lambda^\alpha{}_\beta v^\beta$$

$$\frac{d\bar{v}^\alpha}{d\tau} = \frac{d}{d\tau} \Lambda^\alpha{}_\beta v^\beta = \Lambda^\alpha{}_\beta \frac{dv^\beta}{d\tau}$$

These are called Lorentz vector because they transform according to the Lorentz transformation.

Generalize Newton

$$m \frac{d^2 x^\alpha}{d\tau^2} = f^\alpha$$

This equation is Lorentz invariant in case f^α transforms according to

$$\bar{f}^\alpha = \Lambda^\alpha{}_\beta f^\beta$$

II Special Relativity Results

A. Kinematical

Consider two events and calculate coördinate differences

$$c\Delta\bar{t} = \frac{1}{\sqrt{1-u^2/c^2}}\left(c\Delta t - \frac{u}{c}\Delta x\right) = \gamma\left(c\Delta t - \frac{u}{c}\Delta x\right)$$

$$\Delta\bar{x} = \frac{1}{\sqrt{1-u^2/c^2}}\left(-\frac{u}{c}c\Delta t + \Delta x\right) = \gamma\left(-u\Delta t + \Delta x\right)$$

i) Simultaneity at distant places is not an absolute concept

$$\left.\begin{matrix}\Delta t = 0 \\ \Delta x \neq 0\end{matrix}\right\} \rightarrow \Delta\bar{t} \neq 0$$

ii) Time dilation

Consider two ticks of a clock stationary in the barred frame

$$c\Delta\bar{t} = \gamma\left(c\Delta t - u\Delta x/c\right)$$

$$0 = \gamma\left(-u\Delta t + \Delta x\right) \qquad \rightarrow \Delta x = u\Delta t$$

So: $\Delta\bar{t} = \gamma(1-u^2/c^2)\Delta t$

$$\Delta t = \frac{1}{\sqrt{1-u^2/c^2}}\Delta\bar{t} \qquad \rightarrow \text{Moving clocks run slow}$$

Consider two ticks of a clock stationary in the unbarred frame

$c\Delta\bar{t} = \gamma c \Delta t$

So : $\Delta\bar{t} = \frac{1}{\sqrt{1-u^2/c^2}} \Delta t$ → Moving clocks run slow

iii) Lorentz contraction

Consider a rod which is at rest in the barred frame.

Measure the positions of the end-points such that $\Delta t = 0$.

$c\Delta\bar{t} = -\gamma u \Delta x / c$

$\Delta\bar{x} = \gamma \Delta x$

By construction $\Delta x = l$.

Whatever the value of $\Delta\bar{t}$, by construction $\Delta\bar{x} = \bar{l}$

So : $l = \sqrt{1-u^2/c^2}\, \bar{l}$ → Moving rods are contracted

iv) Adding velocities

Consider a point that moves with velocity v with respect to the barred frame

$$\Delta\bar{x} = v\,\Delta\bar{t}$$

So

$$c\Delta\bar{t} = \gamma(u)\,(c\Delta t - u\Delta x/c)$$
$$v\Delta\bar{t} = \gamma(u)\,(-u\Delta t + \Delta x)$$

$$\rightarrow \quad w = \frac{\Delta x}{\Delta t} = \frac{u+v}{1+uv/c^2}$$

B. Dynamical results.

i) Spatial components

$$m\frac{d^2x^a}{d\tau^2} = f^a$$

Define

$$p^\alpha = m\frac{dx^\alpha}{d\tau} = m\frac{dt}{d\tau}\frac{dx^\alpha}{dt} = \frac{1}{\sqrt{1-v^2/c^2}}\frac{dx^\alpha}{dt}$$

$$(p^\alpha) = \frac{1}{\sqrt{1-v^2/c^2}}\left(c,\ \frac{dx^1}{dt},\ \frac{dx^2}{dt},\ \frac{dx^3}{dt}\right)$$

Then $\quad \dfrac{dp^a}{d\tau} = f^a$

$$\frac{dp^a}{dt} = \sqrt{1-v^2/c^2}\, f^a = F^a$$

$$\frac{d}{dt}\frac{m}{\sqrt{1-v^2/c^2}}\frac{dx^a}{dt} = F^a$$

Generalized Newton.

(ii) Time component, energy

$$E = \int \vec{F}\cdot d\vec{s} = \int \vec{F}\cdot\vec{v}\, dt \quad \rightarrow \quad \frac{dE}{dt} = \vec{F}\cdot\vec{v}$$

Consider

$$\eta_{\alpha\beta}\, dx^\alpha dx^\beta = c^2 d\tau^2$$

$$\eta_{\alpha\beta} \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = c^2$$

$$\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} - \sum_a \frac{dx^a}{d\tau}\frac{dx^a}{d\tau} = c^2$$

Differentiate with respect to t

$$\frac{dx^0}{d\tau}\frac{d}{dt}\frac{dx^0}{d\tau} - \sum_a \frac{dx^a}{d\tau}\frac{d}{dt}\frac{dx^a}{d\tau} = 0$$

$$\frac{dx^0}{d\tau}\frac{dp^0}{dt} - \sum_a \frac{dx^a}{d\tau} F^a = 0$$

$$\frac{dx^0}{dt}\frac{dp^0}{dt} = \sum_a \frac{dx^a}{dt} F^a = \vec{F}\cdot\vec{v}$$

$$\rightarrow \frac{dE}{dt} = c\frac{dp^0}{dt} \quad \rightarrow \quad E = cp^0 = \frac{mc^2}{\sqrt{1-v^2/c^2}}$$

This is probably the most famous equation of physics.

C. Electromagnetic forces

From classical physics we know that forces on particles in magnetic fields are proportional to their velocities.
Let us try to generalize

$$f^{\alpha} = \frac{q}{c} F^{\alpha\beta} \eta_{\beta\gamma} \frac{dx^{\gamma}}{d\tau}$$

The proportionality factor has been written as $F^{\alpha\beta}\eta_{\beta\gamma}$ for convenience
We will show that necessarily

$$F^{\alpha\beta} = - F^{\beta\alpha}$$

Notation

$$(F^{\alpha\beta}) = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

Recall the equations of motion

$$\frac{dp^a}{dt} = F^a = \sqrt{1-v^2/c^2}\, f^a$$

$$F^1 = \sqrt{1-v^2/c^2}\,\frac{q}{c}\left[E_x \eta_{00} \frac{c}{\sqrt{1-v^2/c^2}} - B_z \eta_{22} \frac{\frac{dy}{dt}}{\sqrt{1-v^2/c^2}} + B_y \eta_{33} \frac{\frac{dz}{dt}}{\sqrt{1-v^2/c^2}} \right]$$

$$= \frac{q}{c}\left[c E_x + B_z \frac{dy}{dt} - B_y \frac{dz}{dt} \right]$$

In general

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right)$$ Lorentz force

To conclude

i) we have seen

$$\frac{dx^0}{d\tau}\frac{dp^0}{d\tau} - \sum_a \frac{dx^a}{d\tau}\frac{dp^a}{d\tau} = 0$$

$$\eta_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dp^\beta}{d\tau} = 0$$

So

$$\frac{dx^\alpha}{d\tau}\eta_{\alpha\beta} F^{\beta\gamma}\eta_{\gamma\delta}\frac{dx^\delta}{d\tau} = 0$$

$$\left(\frac{dx^\alpha}{d\tau}\eta_{\alpha\beta}\right) F^{\beta\gamma}\left(\eta_{\gamma\delta}\frac{dx^\delta}{d\tau}\right) = 0 \quad \rightarrow \quad F^{\beta\gamma} = -F^{\gamma\beta}$$

ii) Transformation law for the field tensor

$$\bar{f}^{\alpha} = \Lambda^{\alpha}{}_{\beta} f^{\beta} = \frac{q}{c} \Lambda^{\alpha}{}_{\beta} F^{\beta\gamma} \eta_{\gamma\delta} \frac{dx^{\delta}}{d\tau}$$

$$= \frac{q}{c} \Lambda^{\alpha}{}_{\beta} F^{\beta\gamma} \Lambda^{\mu}{}_{\gamma} \Lambda^{\nu}{}_{\delta} \eta_{\mu\nu} \frac{dx^{\delta}}{d\tau}$$

$$= \frac{q}{c} \Lambda^{\alpha}{}_{\beta} F^{\beta\gamma} \Lambda^{\mu}{}_{\gamma} \eta_{\mu\nu} \frac{d\bar{x}^{\nu}}{d\tau}$$

$$= \frac{q}{c} \bar{F}^{\alpha\mu} \eta_{\mu\nu} \frac{d\bar{x}^{\nu}}{d\tau}$$

$$\longrightarrow \quad \bar{F}^{\alpha\mu} = \Lambda^{\alpha}{}_{\beta} F^{\beta\gamma} \Lambda^{\mu}{}_{\gamma}$$

From this we obtain the transformations of the fields.

III General Relativity; metric theories

A. Principles

i) Eötvös:

In a gravitational field all particles fall equally fast.

Consequently:

In a freely falling frame, free particles move along straight lines.
Using Cartesean coordinates

$$\frac{d^2\xi^a}{dt^2} = 0 .$$

ii) Einstein:

In a freely falling frame Special Relativity holds.

Consequently:

All information about the gravitational field resides in the relation between the freely falling frame and the experimental frame

Remark:

Freely falling frames exist only locally in spacetime

B. Generalised coordinates

i) Freely falling coordinates (ξ^α).

Equation of motion for a free particle

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0$$

Experimental coordinates (x^α)

Equation of motion of a free particle

$$\frac{d}{d\tau}\left[\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right] = 0$$

$$\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{d^2 x^\mu}{d\tau^2} + \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

$$\underbrace{\frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial \xi^\alpha}{\partial x^\mu}}_{\frac{\partial x^\lambda}{\partial x^\mu} = \delta^\lambda_\mu}\frac{d^2 x^\mu}{d\tau^2} + \underbrace{\frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}}_{\Gamma^\lambda_{\mu\nu}}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

$\Gamma^\lambda_{\mu\nu}$ ← Christoffel symbol

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$ Eq. of motion

Line element

$$ds^2 = \eta_{\alpha\beta}\, d\xi^\alpha d\xi^\beta = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} dx^\mu \frac{\partial \xi^\beta}{\partial x^\nu} dx^\nu$$

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

$$g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu}$$

ii) Metric theory

Can we get rid of reference to the freely falling coordinates?

We will demonstrate

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\lambda} \left\{ \frac{\partial g_{\beta\lambda}}{\partial x^\gamma} + \frac{\partial g_{\gamma\lambda}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\lambda} \right\} \qquad \left(g^{\alpha\lambda} g_{\lambda\beta} = \delta^\alpha_\beta \right)$$

So, once we know the metric, we can calculate the Christoffel symbol and write down the equations of motion.

We may therefore conclude that

all information about the gravitational field resides in the metric.

Mathematial details

$$g_{\alpha\beta} = \eta_{\mu\nu} \frac{\partial \xi^\mu}{\partial x^\alpha} \frac{\partial \xi^\nu}{\partial x^\beta}$$

$$\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} = \eta_{\mu\nu} \frac{\partial^2 \xi^\mu}{\partial x^\alpha \partial x^\gamma} \frac{\partial \xi^\nu}{\partial x^\beta} + \eta_{\mu\nu} \frac{\partial \xi^\mu}{\partial x^\alpha} \frac{\partial^2 \xi^\nu}{\partial x^\beta \partial x^\gamma}$$

$$= \eta_{\mu\nu} \frac{\partial^2 \xi^\lambda}{\partial x^\alpha \partial x^\gamma} \delta^\mu_\lambda \frac{\partial \xi^\nu}{\partial x^\beta} + \eta_{\mu\nu} \frac{\partial \xi^\mu}{\partial x^\alpha} \delta^\nu_\lambda \frac{\partial^2 \xi^\lambda}{\partial x^\beta \partial x^\gamma}$$

$$= \eta_{\mu\nu} \underbrace{\frac{\partial^2 \xi^\lambda}{\partial x^\alpha \partial x^\gamma} \frac{\partial x^\delta}{\partial \xi^\lambda}}_{\Gamma^\delta_{\alpha\gamma}} \frac{\partial \xi^\mu}{\partial x^\delta} \frac{\partial \xi^\nu}{\partial x^\beta} + \eta_{\mu\nu} \frac{\partial \xi^\mu}{\partial x^\alpha} \frac{\partial \xi^\nu}{\partial x^\delta} \underbrace{\frac{\partial x^\delta}{\partial \xi^\lambda} \frac{\partial^2 \xi^\lambda}{\partial x^\beta \partial x^\gamma}}_{\Gamma^\delta_{\beta\gamma}}$$

$$\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} = \Gamma^\delta_{\alpha\gamma} g_{\delta\beta} + \Gamma^\delta_{\beta\gamma} g_{\alpha\delta} \qquad (*)$$

By permuting indices

$$\frac{\partial g_{\gamma\beta}}{\partial x^\alpha} = \Gamma^\delta_{\alpha\gamma} g_{\delta\beta} + \Gamma^\delta_{\alpha\beta} g_{\gamma\delta} \qquad (**)$$

$$\frac{\partial g_{\alpha\gamma}}{\partial x^\beta} = \Gamma^\delta_{\alpha\beta} g_{\gamma\delta} + \Gamma^\delta_{\beta\gamma} g_{\alpha\delta} \qquad (***)$$

$(*) + (**) - (***)$: $\dfrac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \dfrac{\partial g_{\gamma\beta}}{\partial x^\alpha} - \dfrac{\partial g_{\alpha\gamma}}{\partial x^\beta} = 2\,\Gamma^\delta_{\alpha\gamma} g_{\delta\beta}$

etc.

IV General Relativity; Results

A. Newtonian limit, gravitational redshift

In the neighborhood of the earth we expect

i) $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ with $|h_{\alpha\beta}| \ll 1$

$$\frac{\partial h_{\alpha\beta}}{\partial x^0} = 0$$

ii) $\dfrac{dx^a}{d\tau} \ll \dfrac{dx^0}{d\tau}$

The equation of motion for a freely falling particle reads

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} = 0$$

Evaluate everything to first order in $h_{\alpha\beta}$ and find

$$\frac{d^2x^a}{dt^2} = -\frac{1}{2}c^2\frac{\partial h_{00}}{\partial x^a}$$

$$\frac{d^2x^a}{dt^2} = -\frac{\partial \Phi}{\partial x^a} \qquad \text{Newton}$$

$$\rightarrow g_{00} = 1 + 2\Phi/c^2$$

Gravitational red shift

Consider two clocks at rest at different positions in a gravitational field.

With every tick of clock A send a photon to clock B.

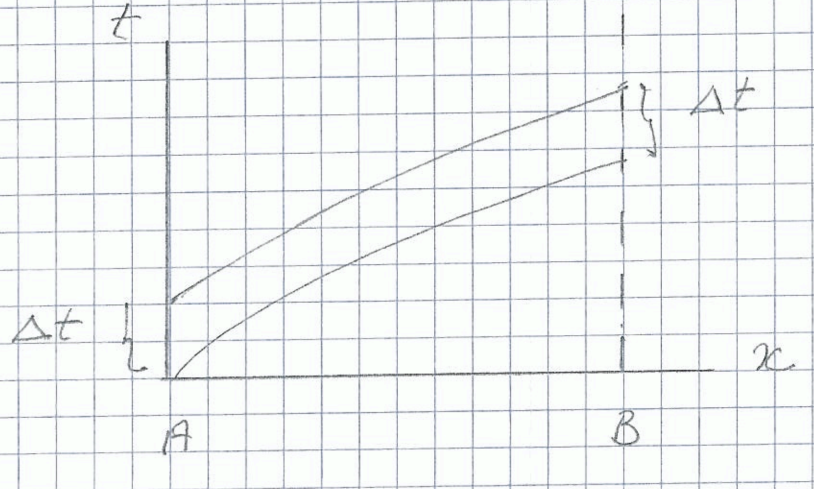
For photons

$$ds^2 = 0$$

$$g_{00}\, c^2\, dt^2 + \sum_a \sum_b g_{ab}\, dx^a\, dx^b = 0$$

$$dt^2 = -\frac{1}{c^2} \sum_a \sum_b \frac{g_{ab}}{g_{00}}\, dx^a\, dx^b \qquad \text{independent of } x^0$$

$$t_a - t_e = \frac{1}{c} \int \sqrt{-\frac{g_{ab}}{g_{00}} \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda}}\, d\lambda \qquad \text{time of flight}$$



The coordinate time difference between the emission of two photons at A is equal to the coordinate time difference between the arrivals of the two photons at B.

The time difference between the two emissions at A is

$$\Delta\tau_A = \sqrt{g_{00}(\vec{r}_A)}\,\Delta t$$

The time difference between the two arrivals at B is

$$\Delta\tau_B = \sqrt{g_{00}(\vec{r}_B)}\,\Delta t$$

Conclusion: $\dfrac{\Delta\tau_B}{\Delta\tau_A} = \sqrt{\dfrac{g_{00}(\vec{r}_B)}{g_{00}(\vec{r}_A)}} = 1 + \frac{1}{2}\left(h_{00}(\vec{r}_B) - h_{00}(\vec{r}_A)\right)$

$$\frac{\Delta\tau_B}{\Delta\tau_A} = 1 + \left(\Phi(\vec{r}_B) - \Phi(\vec{r}_A)\right)/c^2$$

Between Turin and Monte Rosa clocks run at different speeds by about 30 nanoseconds/day

Corrections like this are used in G.P.S.

B. Perihelion shift.

The metric around a heavy spherical mass is given by Schwarzschild (and Droste)

$$ds^2 = B(r)\, c^2 dt^2 - A(r)\, dr^2 - r^2 (d\vartheta^2 + \sin^2\vartheta \; d\varphi^2)$$

$$A(r) = \frac{1}{1 - \frac{r_S}{r}}$$

$$B(r) = 1 - \frac{r_S}{r}$$

$$r_S = \frac{2GM}{c^2}$$

r_S is called the Schwarzschild radius.

Earth : $r_S = 0.886$ cm $= 1.39 * 10^{-9} R_\oplus$

Sun : $r_S = 2.95$ km $= 4.14 * 10^{-6} R_\odot$

Notice that exterior to the sun r_S / r is very small and both $A(r)$ and $B(r)$ are almost equal to unity, i.e. ds^2 is almost equal to the usual special relativistic metric expressed in polar coordinates.

Deviations from Newton's predictions are expected to be small.

Equations of motion

$$\ddot{x}^0 + \frac{B'}{B}\dot{x}^0\dot{r} = 0$$

$$\ddot{r} + \frac{B'}{2A}(\dot{x}^0)^2 + \frac{A'}{2A}\dot{r}^2 - \frac{r}{A}\dot{\vartheta}^2 - \frac{r\sin^2\vartheta}{A}\dot{\varphi}^2 = 0$$

$$\ddot{\vartheta} + \frac{2}{r}\dot{r}\dot{\vartheta} - \sin\vartheta\cos\vartheta\,\dot{\varphi}^2 = 0$$

$$\ddot{\varphi} + \frac{2}{r}\dot{r}\dot{\varphi} + 2\frac{\cos\vartheta}{\sin\vartheta}\dot{\vartheta}\dot{\varphi} = 0$$

Dots indicate differentiations with respect to τ and primes differentiations with respect to r.

All first integrals can easily be obtained

i) $\dot{x}^0 = \dfrac{F}{1 - \frac{r_s}{r}}$ $\qquad F = \text{const.}$

ii) Choose $\vartheta(0) = \pi/2$, $\dot{\vartheta}(0) = 0$

Then

$\vartheta(\tau) = \frac{1}{2}\pi$

iii) $\dot{\varphi} = \dfrac{l}{m r^2}$ $\qquad l = \text{const.}$

iv) $\dot{r} = \frac{1}{\sqrt{A}} \sqrt{\frac{F^2}{B} - \frac{l^2}{m^2 r^2} - c^2}$

The second and third together express conservation of angular momentum.

The fourth may be written as

$$\underbrace{\tfrac{1}{2} m \dot{r}^2 + \frac{l^2}{2mr^2}}_{\text{kinetic}} \underbrace{- \frac{GmM}{r}}_{\text{potential}} \underbrace{- \frac{GMl^2}{mc^2 r^3}}_{\text{relativistic}} = \underbrace{\frac{m}{2}(F^2 - c^2)}_{\text{energy } (E)}$$

We restrict ourselves to calculating the orbit, not the path

$$\frac{dr}{d\varphi} = \frac{\frac{dr}{d\tau}}{\frac{d\varphi}{d\tau}} = \frac{\dot{r}}{\dot{\varphi}}$$

- Neglecting the relativistic term we easily find

$$\frac{1}{r} = \frac{GMm^2}{l^2}\left(1 + e\cos(\varphi - \varphi_p)\right)$$

$$e = \sqrt{1 + \frac{2El^2}{G^2 M^2 m^3}}$$

The radius is minimal when $\varphi = \varphi_p$.

- With relativistic term

Some lengthy algebra boils down to the substitution

$$\frac{1}{r} = \frac{r_a + r_p}{2 r_a r_p} \left\{ 1 + \frac{r_a - r_p}{r_a + r_p} \cos\chi \right\}$$

with r_a and r_p the longest and shortest distance to the sun.

The orbit obeys

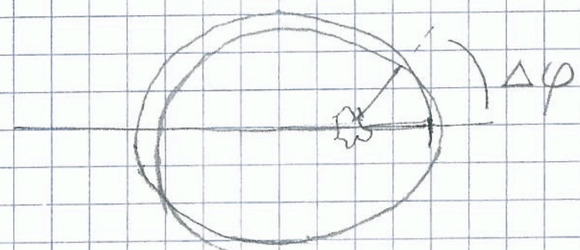
$$\frac{d\varphi}{d\chi} = 1 + \frac{3}{4} r_S \left(\frac{r_a + r_p}{r_a r_p} \right) + \frac{1}{4} r_S \left(\frac{r_a - r_p}{r_a r_p} \right) \cos\chi$$

The radius is periodic in χ.

Integrate

$$\int_0^{2\pi} \frac{d\varphi}{d\chi}\, d\chi = 2\pi + \frac{3}{2}\pi r_S \left(\frac{r_a + r_p}{r_a r_p} \right)$$

$$\longrightarrow \quad \Delta\varphi = \frac{3}{2}\pi r_S \left(\frac{r_a + r_p}{r_a r_p} \right)$$



Exp. Mercury 43.11 seconds/century

Theory Mercury 24.91 seconds/century

V Black holes.

What if $r_S > R$, with R the radius of the heavy mass.
Then we call the object a black hole.
Consider a radially infalling observer with initial conditions

$$r(0) = r_0$$
$$\dot{r}(0) = 0$$

Same analysis as before leads to

$$\dot{x}^0 = \frac{F}{1 - \frac{r_S}{r}}$$

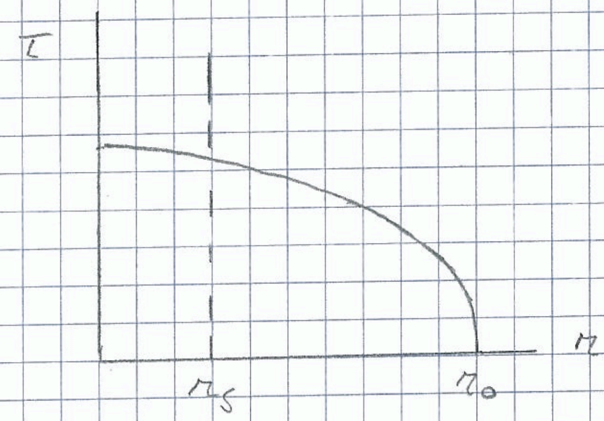
$$\dot{r} = -c\sqrt{\frac{r_S}{r_0}}\sqrt{\frac{r_0}{r} - 1}$$

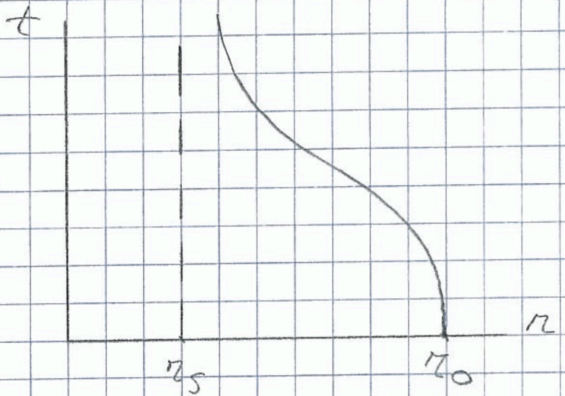
- Eigentime of observer

$$\tau = \int_{r_0}^{r} \frac{1}{\dot{r}}\,dr = \frac{1}{c}\sqrt{\frac{r_0}{r_S}}\left\{\sqrt{r(r_0 - r)} + r_0 \arccos\sqrt{\frac{r}{r_0}}\right\}$$

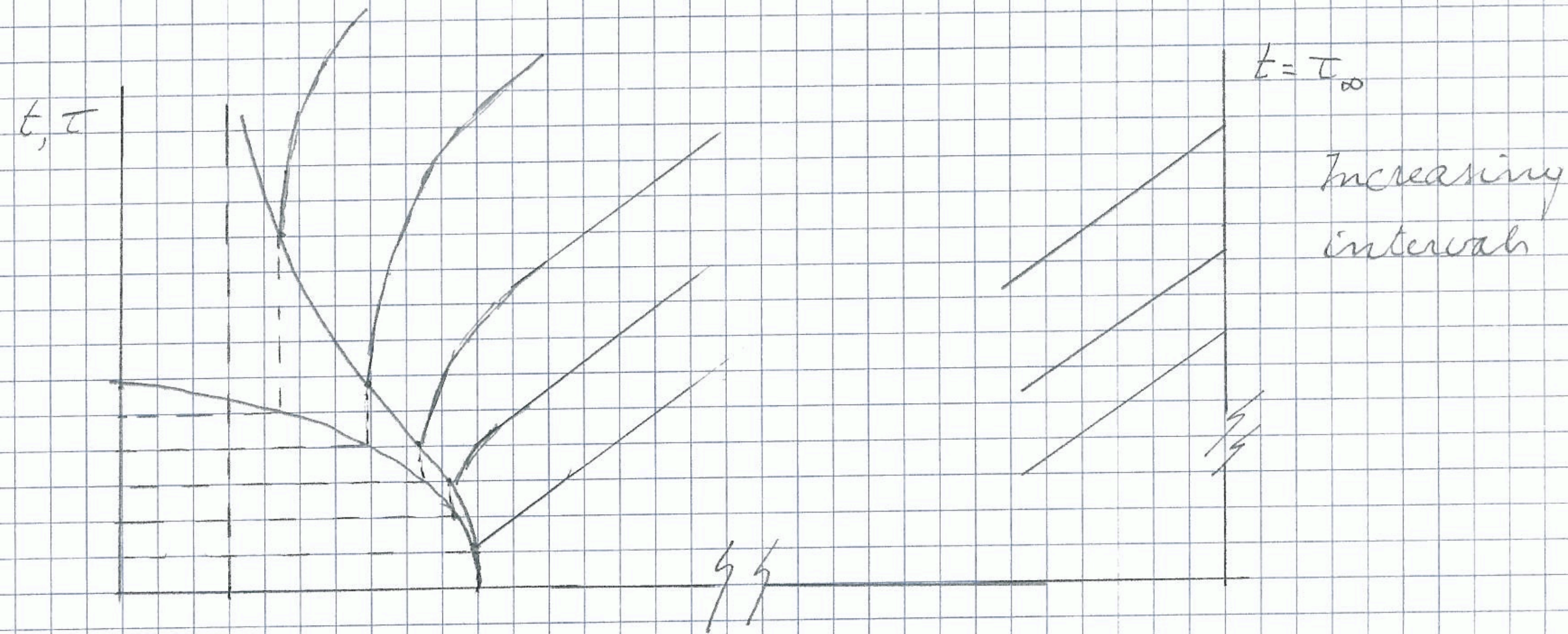
- Coordinate time

$$t = \int_{r_0}^{r} \frac{dt}{dr}\,dr \qquad \text{(see lecture notes)}$$



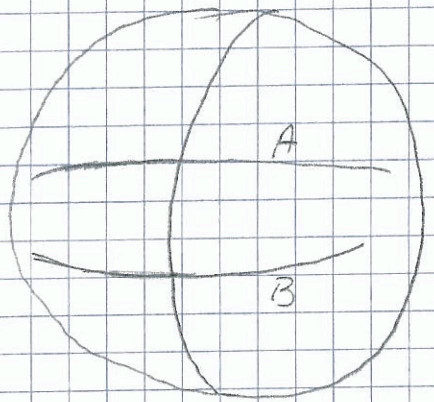


Let the observer send photons with regular intervals on his clock



VI The Einstein equation

Consider two astronauts moving on two great circles around the sun, at the same distance to the sun.
Let the great circles both be perpendicular to a third great circle.



- The astronauts will notice gravity because they approach each other and then separate again etc.
- The astronauts will notice gravity because they move on a curved surface.

→ Gravity = curvature

The vector (ξ^α) between the two astronauts obeys

$$\frac{d^2\xi^\alpha}{d\tau^2} = -R^\alpha{}_{\mu\nu\lambda}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\xi^\lambda$$

$$R^\alpha{}_{\mu\nu\lambda} = \frac{\partial\Gamma^\alpha_{\mu\nu}}{\partial x^\lambda} - \frac{\partial\Gamma^\alpha_{\mu\lambda}}{\partial x^\nu} + \Gamma^\alpha_{\lambda\kappa}\Gamma^\kappa_{\mu\nu} - \Gamma^\alpha_{\nu\kappa}\Gamma^\kappa_{\mu\lambda}$$

- $R^\alpha{}_{\mu\nu\lambda}$ is the curvature tensor
- It governs the dynamics of (ξ^α), resulting from gravity.
- It contains second order derivatives of the metric $(g_{\alpha\beta})$.

Let us now try to generalize Newtons law of gravity

$$\nabla^2\Phi = 4\pi G\rho$$

$$\nabla^2 g_{00} = \frac{8\pi G}{c^4}\rho c^2$$

It turns out that there is really just one possibility

$$R_{\alpha\beta} = -\frac{8\pi G}{c^4}\left(T_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}T\right)$$ Einstein

Here $R_{\alpha\beta} = R^\lambda{}_{\alpha\lambda\beta}$

$T_{\alpha\beta}$ = energy-momentum tensor (mass distribution)

$T = g^{\lambda\mu}T_{\lambda\mu}$